Series I, exercise 1 A pawn starts at the point $(0, 0)$ on the plane and makes a sequence of jumps. If the position of the pawn is (x, y), where $x, y \in \mathbb{Z}$, then after a jump its position is of the form $(x + n, y + k)$, where $n, k \in \mathbb{Z}$ and $|n| + |k| = 5$. What is the smallest number of jumps needed for the pawn to reach the point (2024, 2024)?

Solution Each hop can be described by a displacement vector $< p, q >$ with possible vectors being (up to permutation of coordinates and sign changes) $< 0, 5 >, < 1, 4 >, < 2, 3 >$. The total distance the pawn needs to cover is 4048 with it covering the distance equal to 5 in each step. Because of that the minimal number of jumps cannot be lower that $\lfloor \frac{8048}{5} \rfloor = 809$. Note that after 808 jumps the pawn can reach the position $(2020, 2020)$. Then the remaining distance to cover is 8 meaning the pawn must make at least two more jumps. The jumps in question are (for example) $(4, -1)$ and $(0, 5)$ bringing the minimal number of jumps to 810.

Series I, exercise 2 Prove that for any prime number p there exist at most 2 natural numbers n for which $p2^n + 1$ is a square of a natural number.

Solution Suppose that the number $p2^n + 1$ is a square of some natural number. Since $p2^n + 1$ is odd, then $p2^n + 1 = (2k+1)^2$ for some $k \in \mathbb{N}$. Hence

$$p2^n = 4k^2 + 4k.$$

If $p = 2$, then we have

$$2^{n+1} = 4k(k+1),$$

so

$$2^{n-1} = k(k+1).$$

Therefore, both k and $k+1$ must be powers of 2 which is possible only if $k = 1$. Then

$$2^{n-1} = 2,$$

so $n = 2$ is the only solution.

Now, assume that $p > 2$. Then

$$p2^{n-2} = k(k+1).$$

Of course, p and 2^{n-2} are coprime, and also k and $k+1$ are coprime, so either $k = p$ and $k + 1 = 2^{n-2}$ or $k = 2^{n-2}$ and $k + 1 = p$. Thus, either $p + 1 = 2^{n-2}$ or $2^{n-2} + 1 = p$, and in consequence $2^{n-2} = p+1$ or $2^{n-2} = p - 1$. Finally, for any prime number p there can be only 2 natural numbers n such that $p2^n + 1$ is a square of a natural number.

Series I, exercise 3 A square matrix A is called magic if sums of elements in each row, in each column and in both diagonals are equal. Prove that

$$A = \begin{bmatrix} s+x & s-x+y & s-y \\ s-x-y & s & s+x+y \\ s+y & s+x-y & s-x \end{bmatrix}$$

for some $s, x, y \in \mathbb{R}$. Show that if A is an invertible magic matrix, then A^{-1} is also magic. Moreover, show that A is invertible if and only if $x^2 \neq y^2$.

Solution Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a magic matrix. From this we get

$$a_{22} = 3s - a_{11} - a_{33}$$

$$a_{22} = 3s - a_{13} - a_{31}$$

$$a_{22} = 3s - a_{12} - a_{32}$$

$$a_{22} = 3s - a_{21} - a_{23}$$

Hence $a_{22} = s$. From the above we obtain $a_{11} = s + x$ and $a_{33} = s - x$. Next we get $a_{31} = s + y$ and $a_{13} = s - y$. From $a_{11} + a_{12} + a_{13} = 3s$ we get $a_{12} = s - x + y$ and $a_{11} + a_{21} + a_{31} = 3s$ we obtain $a_{21} = s - x - y$. From the above we get $a_{32} = s + x - y$ and $a_{23} = s + x + y$. We have matrix A in the form

$$A = \begin{bmatrix} s+x & s-x+y & s-y \\ s-x-y & s & s+x+y \\ s+y & s+x-y & s-x \end{bmatrix}$$

The determinant of the matrix A is $9s(x^2 - y^2)$, so if $x^2 = y^2$ then the matrix A is singular; otherwise is regular.

Now suppose that A is regular. Then

$$A^{-1} = \begin{bmatrix} \frac{x^2-y^2-3sy}{9s(x^2-y^2)} & \frac{x+y-3s}{9s(x+y)} & \frac{x^2-y^2+3sx}{9s(x^2-y^2)} \\ \frac{x-y+3s}{9s(x-y)} & \frac{1}{9s} & \frac{x-y-3s}{9s(x-y)} \\ \frac{x^2-y^2-3sx}{9s(x^2-y^2)} & \frac{x+y+3s}{9s(x+y)} & \frac{x^2-y^2+3sy}{9s(x^2-y^2)} \end{bmatrix}$$

and it is easy to check that this matrix is also magic.