Recent results in conformal foliation theory

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Plan of the talk

1. History of foliations with geometrical leaves
2. Conformal tools
3. Canal foliations
4. Umbilical foliations
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1. History of foliations with geometrical leaves
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A $C^r$ codimension $q$ **foliation** of an $n$–dimensional manifold $M$ is a decomposition of $M$ into it $p = (n - q)$–dimensional submanifolds (leaves) looking locally as product $\mathbb{R}^p \times \mathbb{R}^q$ provided that change of such ”product” maps is $C^r$.

Generally in foliation theory we study foliations on **compact** manifolds but leaves are not necessary compact.

One of the first known foliations is the **Reeb foliation** of $S^3$ containing torus and planes spiralling on this torus from both sides.
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Reeb foliation

Figure: Reeb component inside torus
Totally geodesic and totally umbilical

Definition

A submanifold $L$ of a Riemannian manifold $M$ is called **totally geodesic** if at any point $p$ its shape operator $A_p$ vanishes. If at any $p \in L$ the shape operator is a homothety i.e. $A_p = \lambda(p) I_d$ then $L$ is **totally umbilical**.

We say that a foliation is **totally geodesic** (resp. **totally umbilical**) if all its leaves have this property.
Non–existence of totally geodesic foliations

Theorem (Brito, Ghys, Walczak, Zeghib 1981–97)

There is no $C^r$ codimension $q$ totally geodesic codimension $q$ foliation on a compact hyperbolic $n$–manifold for any $r$, $q$ and $n$.

- Totally geodesic foliations on $\mathbb{R}^n$ exist (parallel hyperplanes) and they generate those on tori.
- $S^n$ does not admit totally geodesic foliations of codimension 1 of purely topological reasons.
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Non–existence of geometrically defined foliations

A compact hyperbolic $n$–manifold does not admit a foliation which is

- **Riemannian**: Walschap 1998
  Riemannian = leaves are locally equidistant

- **quasi–isometric**: Fenley 1992 for $n = 3$, $q = 1$ quasi–isometric
  = inclusion of leaves in the universal cover are quasi–isometric embedding i.e.

  $$\frac{1}{\lambda}d(x, x') - \epsilon \leq d(f(x), f(x')) \leq \lambda d(x, x') + \epsilon$$

  with $\lambda, \epsilon$ uniformly bounded

- **totally umbilical**: Langevin–Walczak 2008 for $q = 1$. 

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Recent results in conformal foliation theory
Conformal invariants for surface

For a surface $S \subset \mathbb{R}^3 \subset S^3$ with non-zero Gaussian curvature and free of umbilics i.e. its principal curvatures $k_1$ and $k_2$ are distinct everywhere we denote by $X_1$ and $X_2$ principal directions, and

$$\mu = \frac{k_1 - k_2}{2}.$$ 

Then define

- **principal conformal curvatures**

  $$\theta_1 = \frac{1}{\mu^2} X_1(k_1), \quad \theta_2 = \frac{1}{\mu^2} X_2(k_2)$$

- **principal conformal vector fields**

  $$\xi_1 = \frac{1}{k_1} X_1, \quad \xi_2 = \frac{1}{k_2} X_2$$
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Conformal invariants for surface

and finally

- the Bryant invariant

$$\psi = \frac{1}{\mu^3} (\triangle H + 2\mu^2 H) + \frac{1}{2} (\xi_1^2 - \xi_2^2 + \xi_1(\theta_1) + \xi_2(\theta_2))$$

Here $H$ is the mean curvature of $S$.

Quantities $\theta_1, \theta_2, \xi_1, \xi_2, \psi$ define a surface up to a conformal transformation from the group $\text{M"{o}b}_3 \simeq O^+(3,1)$ generated by inversions in 2–spheres.
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**Examples of conformal surfaces**

- **canal surface**: $\theta_1 = 0$
  This is an envelope of 1-parameter family of 2-spheres which intersect each other if they are close enough. Thus a canal surface is made of a family of **characteristic circles**.

- **Dupin cyclide**: $\theta_1 = \theta_2 = 0$
  On a Dupin cyclide there are two families of characteristic circles.

  The Bryant invariant distinct Dupin cyclides: $|\Psi| > 2$ is for cones, $|\Psi| = 2$ for cylinders, and $|\Psi| < 2$ is tori.
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Canal surface

Figure: Canal surface with its characteristic circles
Dupin cyclides

Figure: Three types of Dupin cyclides
A compact 3–manifold with constant non-zero sectional curvature does not admit a foliation which is

- **Dupin**: Langevin–Walczak 2008
- **of constant conformal invariants (CCI)**: Bartoszek–Walczak 2008

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Theorem (Langevin–Walczak 2010)

Any foliation of the sphere $S^3$ by canal surfaces is Reeb foliation with toral leaf being a Dupin cyclide or is obtained from such Reeb foliation inserting zone $T^2 \times [0, 1]$ consisting of toral or cylindrical leaves.

The zone contains

- finite number of essential zones where cylindrical leaves accumulate on two boundary tori inducing different orientations,
- finite or countable number of spiralling components where cylindrical leaves accumulate on two boundary tori inducing the same orientation.
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Figure: Two types of essential zones
Spiralling component

Figure: Two types of spiralling component
A gridded structure on a 3–manifold is a continuous orientable foliation or sub-foliation by circles possibly with isolated singularities.
If a gridded structure is piecewise continuous on compact saturated submanifolds with sudden discontinuities along finitely many compact leaves which will be analogues of Dupin cyclides then we call it a topological canal foliations.

Geometric canal foliations are special cases of topological canals.
Griddled structures

Figure: Canonical griddled structures
Manifolds admitting topological canal foliations

Theorem (Hector–Langevin–Walczak 2016 preprint)

A closed (i.e. compact without boundary) 3–manifold $M$ admits a topological canal foliation iff it is one of the following

1. $M = S^3$
2. $M = T^3$
3. $M = S^2 \times S^1$
4. $M$ is a lens space
5. $M$ is $S^1$ bundle over $T^2$.

Lens space = two glued solid along closed geodesics

Corollary

There is no canal foliation on any hyperbolic 3–manifold.
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Corollary

There is no canal foliation on any hyperbolic 3–manifold.
On compact hyperbolic 3–manifold there is no codimension 1 foliations which are

- totally geodesic
- totally umbilical
- Riemannian
- Dupin
- CCI
- canal
- quasi–isometric

Maybe some combination of canal and umbilical is a promise or only noncompact case remains?
Any geometry?

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Theorem (Ferus 1973)

Any codimension 1 totally geodesic $C^2$ foliation of $\mathbb{H}^n$ is orthogonal to a curve $\mathbb{R} \to \mathbb{H}^n$ of geodesic curvature $\leq 1$ and any such foliation appears in the above way.
Figure: Totally geodesic foliation of $\mathbb{H}^n$ which is orthogonal to a geodesic.
In $\mathbb{R}^{n+2}$ consider Lorentz form

$$\langle x|y \rangle = -x_0y_0 + x_1y_1 + \ldots + x_{n+1}y_{n+1}$$

and sets $\mathcal{L}: \langle x|x \rangle = 0$ — light cone

$\Lambda^{n+1}: \langle x|x \rangle = 1$ — de Sitter space

$S^n_\infty: \langle x|x \rangle = 0$ and $x_0 = 1$ — sphere at infinity
De Sitter space

Figure: Light cone and de Sitter space
De Sitter space $\Lambda^{n+1}$ is in one-to-one correspondence with the set of all oriented $(n-1)$–spheres on $S^n$.

For $\sigma \in \Lambda^{n+1}$ its corresponding sphere is

$$\Sigma = \sigma^\perp \cap S^n_\infty$$

Conversely, if $\Sigma$ is $(n-1)$–dimensional sphere in $S^n$ of geodesic curvature $k_g$, $m \in \Sigma$ and $n$ is unit normal to $\Sigma$ with respect to $S^n$ then

$$\sigma = k_g m + n$$
De Sitter space vs. spheres

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De Sitter space vs. spheres

Figure: Spheres in de Sitter space
In the space $\Lambda^4$ of 2-spheres

- canal surfaces are space-like curves
- Dupin cyclides are sections by space-like affine 2-planes
In $\mathbb{H}^n$ they are three candidates for leaves of totally umbilical foliations:

- totally geodesic hypersurfaces isometric to $\mathbb{H}^{n-1}$,
- hyperspheres (equidistant from totally geodesic) isometric to $\mathbb{H}^{n-1}$ of constant curvature between $-1$ and $0$,
- horospheres isometric to $\mathbb{R}^{n-1}$.

All of them are parts of $(n - 1)$-spheres so are visible in $\Lambda^{n+1}$. Thus totally a umbilical foliation of $\mathbb{H}^n$ is there too.
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Theorem (Cz-Langevin 2013)

Any codimension 1 totally geodesic foliations of $\mathbb{H}^n$ appears as an unbounded time-or-light–like curve in de Sitter space $\Lambda^{n+1}$. This curve is contained in its subspace $\Sigma^\perp \Lambda^n$ where $\Sigma$ is the ball model of $\mathbb{H}^n$. 
Pencils of spheres

For two given spheres depending on its intersection we have three types of pencils

- **Poncelet** if they disjoint
- **tangent** for tangent
- **intersecting** if they intersect

If the spheres intersect then we attach to them one intersecting pencil and the family of tangent pencils.
One pencil
Two pencils

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Three pencils
Three pencils with their vectors

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Recent results in conformal foliation theory
Shadok cone

Definition

**Shadok cone** at $\sigma \in \Lambda^{n+1}$ such that $|\langle \sigma | \sigma_{\infty} \rangle| \leq 1$ is the set $Sh_\sigma \subset T_\sigma(\Lambda^{n+1})$ which is the union of local time cone and convex hull of vector $\nu$ tangent to the sharing boundary pencil and light vectors orthogonal to $\nu$. 
Shadok cone

Recent results in conformal foliation theory
Conformal classification of totally umbilical foliations in hyperbolic space

Theorem (Cz–Langevin, still in progress)

Every totally umbilical foliation of $\mathbb{H}^n$ modelled on a oriented sphere $\Sigma_{\infty} \subset S^n$ is represented by a curve $\Gamma : \mathbb{R} \to \Lambda^{n+1}$ included in the band between $\sigma + \sigma \perp$ and $-\sigma + \sigma \perp$ and satisfying condition

$$\Gamma'(t) \in Sh_{\Gamma(t)}.$$