Foliations with special geometric properties

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Part 1

Manifolds, submanifolds and foliations
**Definition 1.** Let $M$ be a topological Hausdorff space. We say that $M$ is an $m$–dimensional **manifold** of class $C^r$ if there is a family of open sets $\{U_\alpha\}$ covering $M$ and a family of homeomorphisms (called **maps**) $\varphi_\alpha : U_\alpha \to V_\alpha$ with $V_\alpha$ open in $\mathbb{R}^m$ such that for any $\alpha, \beta$ a map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is of class $C^r$.

**Example 1.** $\mathbb{R}^n$ is an $n$–dimensional manifold.

$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is an $n$–dimensional manifold. Two maps covering $S^n$ are stereographic projection from the poles $N$ and $S$. 
Definition 2. A **tangent vector** to a manifold $M$ at a point $p$ is a linear real function $v$ on the space of smooth functions in neighbourhoods of $p$ on $M$ which acts as derivative i.e.

$$v(f_1 \cdot f_2) = f_1(p)v(f_2) + f_2(p)v(f_1)$$

A collection of all jets of vectors tangent to $M$ at $p$ is its **tangent space** $T_p(M)$ — linear space of dimension $m$.

A **tangent bundle** $TM$ is a vector bundle of tangent spaces over $M$. $TM$ is a manifold of dimension $2m$.

**Example 2.** $T_p(S^n) = p^\perp$ is a hyperplane in $\mathbb{R}^{n+1}$. 
Definition 3. A **vector field** on $M$ is a section of the tangent bundle $TM$.

For two differentiable vector fields $X$, $Y$ on $M$ we define their **Lie bracket** as a vector field

$$[X,Y](f) = X(Yf) - Y(Xf)$$

for any function $f$ on $M$.

**Example 3.** On $S^n$ with $n$ even there is no nonzero smooth (or continuous) vector fields.

In $\mathbb{R}^n$ coordinate vector fields commute i.e. $\left[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$
Definition 4. We say that \( N \subset M \) is its \( n \)-dimensional submanifold, \( n \leq m \), if \( N \) is an image of \( n \)-dimensional manifold under homeomorphism which is immersive i.e. rank of the derivative is \( n \) at every point.

Example 4. \( S^n \) is an \( n \)-dimensional submanifold of \( \mathbb{R}^{n+1} \).

Whitney theorem states that any manifold \( M \) there is \( k \) such that \( M \) is a submanifold of \( \mathbb{R}^k \).
Definition 5. Let $M$ be an $m$–manifold. A foliated chart of codimension $q$ is a pair $(U, \varphi)$, where $U$ is open in $M$ an $\varphi$ is a diffeomorphism of $U$ onto a product of $p = (m - q)$–dimensional neighbourhood $B_\tau$ and $q$–dimensional rectangular neighborhood $B_{\triangleleft}$. A plaque of this foliated chart is a set of form $\varphi^{-1}(B_\tau \times \{y\})$.

A foliation on $M$ of class $C^r$ $(r = 0, 1, \ldots, \infty, \omega)$ and codimension $q$ is defined by $C^r$ atlas consisting foliated charts such that the transition transformation $\varphi_\beta \circ \varphi_\alpha^{-1}$ maps horizontal levels in $\varphi_\alpha(U_\alpha)$ into horizontal levels in $\varphi_\beta(U_\beta)$ for any $\alpha, \beta$.

Leaves of the foliation are connected unions of plaques.
Example 5. • Cartesian product $M \times S^1$ with leaves diffeomorphic to $M$.

• Submersion $f : M \to B$ with leaves which are connected components of nonempty level sets of $f$.

• Linear foliation of tori $T^2 = \mathbb{R}^2/\mathbb{Z}^2$; if the slope is rational leaves are diffeomorphic to $S^1$, if not — to $\mathbb{R}$. 
Reeb foliation of $S^3$

1. Consider submersion $[-1, 1] \times \mathbb{R} \ni (x, y) \mapsto (x^2 - 1)e^y \in \mathbb{R}$ generating a foliation of the infinite strip. Rotate it around $y$–axis to obtain a foliation $\mathcal{F}_0$ of the solid closed cylinder $D^2 \times \mathbb{R}$ invariant under the group $\Gamma$ of vertical translations of $2\pi$ multiplicities.

2. Interior of the solid tori $D^2 \times S^1$ is thus foliated via $\mathcal{F} = \mathcal{F}_0/\Gamma$ (Reeb component) by leaves diffeomorphic to $\mathbb{R}^2$ which accumulate on $T^2$.

3. $S^3 = \{x \mid x_1^2 + \ldots + x_4^2 = 1\}$ is a union of two solid tori with the common boundary $T^2 = \{x \mid x_1^2 + x_2^2 = \frac{1}{2}, x_3^2 + x_4^2 = \frac{1}{2}\}$. We foliate $S^3$ taking two copies of $\mathcal{F}$. 
Theorem 6. (Frobenius) A $k$–dimensional plane distribution $E$ on a manifold $M$ is tangent to a foliation on $M$ iff $E$ is involutive i.e. for any $X, Y \in E$ we have $[X, Y] \in E$.

Theorem 7. (Denjoy) There is a $C^1$ foliation of tori $T^2$ which is not homeomorphic to any $C^2$ foliation of $T^2$.

Theorem 8. (Haefliger) There is no real analytic (i.e. $C^\omega$) foliation of codimension 1 on a closed manifold with finite fundamental group.
**Theorem 9. (Novikov)** Any codimension 1 foliation of class $C^2$ on $S^3$ has a toral leaf with Reeb component inside it.

**Theorem 10. (Thurston)** On any closed manifold of Euler characteristic 0 there is a codimension 1 foliation. In particular any closed 3–manifold admits a codimension 1 foliation.

**Theorem 11. (Cantwell, Conlon)** Every surface (not necessary compact) is a leaf of some foliation on a compact manifold.
**Definition 12.** A **Riemannian metric** on a manifold $M$ is a symmetric and positive $(0,2)$ tensor $\langle ., . \rangle$ i.e. a smooth assigning to any point $p \in M$ an inner product on $T_p(M)$.

On a Riemannian manifold $(M, \langle ., . \rangle)$ we introduce a **Levi–Civita connection** $\nabla : (X, Y) \mapsto \nabla_X Y$ on pairs of vector fields such that for any vector field $Z$ on $M$

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle$$

$$+ \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle$$

**Example 12.** On $\mathbb{R}^n$ the standard inner product induces the co-variant derivative as Levi–Civita connection.
Definition 13. On a Riemannian manifold $M$ a $(1,3)$ tensor $R$ given by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is called a curvature tensor.

A sectional curvature of $M$ at point $p$ in direction of 2–dimensional plane $\subset T_p(M)$ generated by two orthonormal vectors $u, v$ is a number $K(u,v) = \langle R(u,v)v, u \rangle$.

Example 13. The Euclidean space $\mathbb{R}^n$ has constant sectional curvature 0 (i.e. at any point any sectional curvature is 0). The sphere $S^n$ is of constant curvature 1.

If a Riemannian manifold has constant curvature $\kappa$ then

$$R(X,Y)Z = \kappa (\langle Y, Z \rangle X - \langle X, Z \rangle Y)$$
Definition 14. A geodesic on a Riemannian manifold $M$ is a function $c : I \to M$ on an interval $I \subset \mathbb{R}$ satisfying a condition $\nabla_{\dot{c}} \dot{c} = 0$, where $\dot{c}$ denotes the tangent field to $c$.

A geodesic curvature of a curve $\gamma$ parametrized by arc–length on $M$ is a function $k_g(\gamma) = \left\| \nabla_{\dot{\gamma}} \dot{\gamma} \right\|$.

Example 14. In $\mathbb{R}^n$ straight lines contain images of geodesics. On $S^n$ these images are contained in great circles.
**Definition 15.** Let $L$ be a submanifold of a Riemannian manifold $M$ and $\nu$ denotes a unit normal vector field to $L$. A **shape operator** of the submanifold $L$ at $p \in L$ with respect to $\nu(p)$ is a linear self–adjoint endomorphism $A : T_p(L) \to T_p(L)$ given by $A(u) = (\nabla_u \nu) ^\perp$.

Eigenvalues of $A$ are called **principal curvatures** of $L$ at $p$.

**Definition 16.** A submanifold is called **totally geodesic** if its shape operator vanishes at any point.

If a submanifold $L$ has the shape operator proportional to the identity at any point then $L$ is **totally umbilical**.
For further reading:

A. Candel, L. Conlon, *Foliations I, II*

T. Sakai, *Riemannian Geometry*