Mathematical basis of logistics
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Chapter 1

Functions

The concept of function is often used in economics, logistics and computer science. For example, your tax rate is a function of your total earnings, the production cost is a function of cost of materials and goods, time of running algorithm is a function of input data. Now we formulate the definition and properties of a function.

1.1 Function Notation and Properties

Definition 1.1. The mapping $f$ the set $X$ into $Y$ is called the function, if each element of the set $X$ is assigned to exactly one element of the set $Y$. The set $X$ is called domain and $Y$ is called range.

Elements of $X$ are called arguments and assigned them elements of $Y$ - values of the function. In general, when we write $y = f(x)$ we mean that $y$ is the value associated with $x$. Sometimes, we use the notation $D_f$ as the domain of a function $f$.

Remark 1.1. $f : X \rightarrow \mathbb{R}$ are called real(-valued) functions.

Usually, the real function is defined by the formula. e.g.

$$f(x) = 2x^2 + 3x - 4 \text{ for } x \in \mathbb{R}$$

Example 1.1. Given the function defined by the equation $f(x) = 2x^2 + 3x - 4$. Determine $f(0)$, $f(-1)$, $f(2x)$. 

7
Solution.

\[ f(0) = 2 \cdot 0^2 + 3 \cdot 0 - 4 = -4 \]
\[ f(-1) = 2 \cdot (-1)^2 + 3 \cdot (-1) - 4 = -5 \]
\[ f(2x) = 2 \cdot (2x)^2 + 3 \cdot (2x) - 4 = 8x^2 + 6x - 4 \]

Remark 1.2. When the formula of the real function has not been determined domain, by the domain of the function we take the set of all real numbers for which the function has a numerical sense.

Example 1.2. Domain of function \( f(x) = 2x^2 + 3x - 4 \) is the set \( \mathbb{R} \).

Usually, the domain of function has an infinite number of elements. Since it is impossible to list the elements of the function, we could instead describe this function by its graph.

Graph of the function \( f(x) = 2x^2 + 3x - 4 \)

Example 1.3. Determine the domain the function defined by the formula

\[ f(x) = \frac{x^2 + 1}{\sqrt{-x^2 + 5x + 6}} + \sqrt{x} \]

Solution. To determine the domain we must consider two cases:
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1. \(-x^2 + 5x + 6 > 0 \iff x \in (-1, 6)\)

2. \(x \geq 0 \iff x \in (0, \infty)\)

therefore, if we find the common part of this two intervals, then we determine the domain.

Hence, the domain is \(X = (0, 6)\).

**Definition 1.2.** A function \(f : X \rightarrow Y\) is called injective, if for any, different arguments \(x_1, x_2 \in X\) and \(x_1 \neq x_2\) we have

\[f(x_1) \neq f(x_2)\]

(for different arguments we have different values of function).

**Remark 1.3.** In applications we often use different but equivalent definition: if \(f(x_1) = f(x_2)\), then \(x_1 = x_2\).

Notation of an injective function

\[f : X \xrightarrow{1-1} Y\]

**Example 1.4.** Examine the injective of function \(f(x) = x^3 + 2, x \in \mathbb{R}\).

**Solution.** For any \(x_1, x_2 \in \mathbb{R}\), we have

\[f(x_1) = f(x_2) \Rightarrow x_1^3 + 2 = x_2^3 + 2\]

\[\Rightarrow x_1^3 = x_2^3\]

\[\Rightarrow x_1 = x_2\]

**Definition 1.3.** A function \(f : X \rightarrow \mathbb{R}\) is called bounded from above(or bounded from below), if there exists a \(M \in \mathbb{R}\), such that, for any \(x \in X\), we have

\[f(x) \leq M\]

(or there exists a \(m \in \mathbb{R}\), such that, for any \(x \in X\), we have

\[f(x) \geq m\]

).
Example 1.5. Because of

\[-1 \leq \sin x \leq 1\]

the function \( f(x) = \sin x \) is bounded from above and from below. Shortly we say that \( f \) is bounded.

Let \( x_1, x_2 \in X \)

Definition 1.4. Function \( f : X \rightarrow R \) is called

- **increasing** if for all \( x_1, x_2 \in X \) such that \( x_1 < x_2 \) we have

  \[ f(x_1) < f(x_2) \]

- **decreasing** if for all \( x_1, x_2 \in X \) such that \( x_1 < x_2 \) we have

  \[ f(x_1) > f(x_2) \]

- **constant** if for all \( x_1, x_2 \in X \) we have

  \[ f(x_1) = f(x_2) \]

- **nondecreasing** if for all \( x_1, x_2 \in X \) such that \( x_1 < x_2 \) we have

  \[ f(x_1) \leq f(x_2) \]

- **nonincreasing** if for all \( x_1, x_2 \in X \) such that \( x_1 < x_2 \) we have

  \[ f(x_1) \geq f(x_2) \]
Example 1.6.

- function \( g(x) = 2x - 3 \) is increasing
- function \( f(x) = x^2 \) is increasing in the interval \((0, \infty)\) and is decreasing in the interval \((-\infty, 0)\).

For the functions can be define arithmetic operations.

**Definition 1.5.** Let \( f, g : X \to \mathbb{R} \). The sum of functions is called function \( f(x) + g(x) \); difference \( f(x) - g(x) \); product of two functions \( f(x) \cdot g(x) \) and quotient \( \frac{f(x)}{g(x)} \) in case of \( g(x) \neq 0 \).

**Example 1.7.** Let \( f(x) = x^2 + 1 \) and \( g(x) = 2x - 1 \).

- the sum of function \( f \) and \( g \): \( f(x) + g(x) = x^2 + 1 + 2x - 1 = x^2 + 2x \)
- the difference of function \( f \) and \( g \): \( f(x) - g(x) = x^2 + 1 - 2x + 1 = x^2 - 2x + 2 \)
- the product of function \( f \) and \( g \): \( f(x) \cdot g(x) = (x^2 + 1)(2x - 1) = 2x^3 - x^2 + 2x - 1 \)
- the quotient of function \( f \) and \( g \): \( \frac{f(x)}{g(x)} = \frac{x^2+1}{2x-1} \) for \( x \neq \frac{1}{2} \)

**Definition 1.6.** Let \( g : X \to Y \) and \( f : Y \to Z \). The superposition of functions \( f \) and \( g \) we called the function \( h : X \to Z \) where

\[
h(x) = f(g(x)) \text{ for all } x \in X
\]
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The function \( f \) we called the outer function and \( g \) the inner function.

**Example 1.8.** Let \( f(x) = x^2 \) and \( g(x) = 2x - 1 \). Then

\[
\begin{align*}
    f(g(x)) &= f(2x - 1) = (2x - 1)^2 \\
    g(f(x)) &= g(x^2) = 2x^2 - 1
\end{align*}
\]

It is easy to notice, that

\[
    f(g(x)) \neq g(f(x))
\]

1.2 Economic Functions

1.2.1 Cost function

Let

- \( x \) the amount of manufactured product
- \( C(x) \) the total cost of production volumes \( x \) of the product
- \( R(x) \) revenue on the sale amount \( x \) of the product
- \( P(x) \) profit on the sale amount of \( x \) of the product

The relationship between cost, revenue and profit is of the form

\[
P(x) = R(x) - C(x)
\]

Production costs of \( x \) goods are divided into: fixed costs, independent of the volume of production (e.g., investments in technical equipment) - are denoted by \( k \) and variable costs that depend on the production - are denoted by \( v \). Then a linear cost function, the production amount of \( x \) of the product is of the form

\[
    C(x) = v \cdot x + k
\]

The company sells a product at a price of \( p \) PLN. The revenue function of amount of \( x \) of the product is of the form

\[
    R(x) = p \cdot x
\]
Example 1.9. The company produces ice cream. Every month, they have the fixed costs in the amount of 750 PLN, and produce a single knob costs 0.20PLN. Price knob is 1.5PLN. How many knobs must produce and sell this company to revenue to balances the costs (in month)?

Solution. We have

- fixed costs \( k = 750\) PLN
- variable costs \( v = 0.2\) PLN
- product price \( p = 1.5\) PLN

The cost function is of the form

\[
C(x) = vx + k = 0.2x + 750
\]

The revenue function is of the form

\[
R(x) = 1.5x
\]

When the revenue balances the the cost, the profit is equal zero

\[
P(x) = 0
\]

Hence

\[
P(x) = R(x) - C(x) = 0 \Rightarrow U(x) = K(x)
\]

\[
P(x) = R(x) - C(x) = 1.3x - 750
\]

So

\[
P(x) = 0 \Leftrightarrow 1.3x - 750 = 0 \Leftrightarrow x = 577
\]

Because of

\[
R(577) = 1.5 \cdot 577 = 865.5\text{PLN}
\]

the company must produce 577 knobs of ice cream and sell them for 865.5PLN to offset production costs.
1.2.2 Tornquist functions

Tornquist functions are simple models of the purchase which depend on income. Let \( x \) be an income. All goods we can divide

- basic goods (flour, cereals, bread, potatoes, tea, etc.)
- superior goods (eg washing machines, refrigerators, televisions, furniture, etc.)
- luxury goods (jewelry, international travel, luxury cars, etc.)

The I kind Törnquist function is of the form

\[
f(x) = a \cdot \frac{x}{x + b}
\]
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The II kind Törnquist function is of the form

\[ f(x) = a \cdot \frac{x - c}{x + b} \]

where \( c \) is a minimal income. The domain of the function is \([c, \infty)\).

The III kind Törnquist function is of the form

\[ f(x) = ax \cdot \frac{x - c}{x + b} \]

where \( c \) is a minimal income. The domain of the function is \([c, \infty)\).
1.2.3 Future value function

Suppose that a sum of money $P_v$ is credited to the account. We know an annual rate $r$, the number of compounding periods per year $n$ and $t$ the number of years. The future value function $F_v$ of our money is given by the formula

$$F_v = P_v \left(1 + \frac{r}{n}\right)^{nt}$$

**Example 1.10.** Suppose 5000PLN is invested at an annual interest rate of $r = 10\%$. Compute the future value after 2 years if the interest is compounded annually, semi annually, quarterly, monthly, daily, hourly, every minute?

**Solution.** We have $t = 2$, $P_v = 5000$

- annually: $n = 1$
  
  $$F_v = 5000 \left(1 + \frac{0.1}{1}\right)^{1 \cdot 2} = 6050$$

- every half year: $n = 2$
  
  $$F_v = 5000 \left(1 + \frac{0.1}{2}\right)^{2 \cdot 2} = 6077.53$$

- quarterly: $n = 4$
  
  $$F_v = 5000 \left(1 + \frac{0.1}{4}\right)^{4 \cdot 2} = 6092.01$$
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• monthly: \( n = 12 \)

\[
F_v = 5000 \left(1 + \frac{0.1}{12}\right)^{2 \cdot 12} = 6101.95
\]

• daily: \( n = 365 \)

\[
F_v = 5000 \left(1 + \frac{0.1}{365}\right)^{365 \cdot 2} = 6106.84
\]

• hourly: \( n = 365 \cdot 24 = 8760 \)

\[
F_v = 5000 \left(1 + \frac{0.1}{8760}\right)^{8760^2} = 6107.006
\]

• every minute: \( n = 365 \cdot 24 \cdot 60 = 525600 \)

\[
F_v = 5000 \left(1 + \frac{0.1}{525600}\right)^{525600^2} = 6107.013
\]

Note that as \( n \) increases, the future value increases, moreover, as \( n \) gets larger the accumulation appears to increase much more slowly, almost becoming fixed. In fact, there is almost no difference between getting interest hourly or every minute. What happens to the accumulation if interest is given continuously, that is, at every instant of time? The future value function is of the form

\[
F_v = P_v e^{rt}
\]

where \( e \) is irrational number, and is one of the most important numbers in mathematics.

\[
e = 2.7182818...
\]

It is often called Euler’s number.

**Remark 1.4.** The number \( e \) is a value that we get after a year, when we put one penny, with an interest rate one, and assuming that the the bank uses the continued capitalization of interest.

### 1.3 Limits

The concept of the limit in calculus is very important. It describes what happens to a function as a particular value is approached.
**Definition 1.7.** \( \lim_{x \to x_0} f(x) = g \) is read the limit of function \( f(x) \) as \( x \) approaches \( x_0 \) is \( g \), (when it exists), is the unique number that the values of functions \( f \) are very near \( g \), when the arguments are very close to \( x_0 \), either just to its left or just to its right.

**Example 1.11.** Find

\[
\lim_{x \to 2} (x + 3)
\]

**Solution.** We want to find the value of function \( f(x) = x + 3 \) that is very near, when \( x \) is very near 2.

\[
\begin{align*}
  f(1.99) &= 4.99 \\
  f(1.999) &= 4.9999 \\
  f(2.01) &= 5.01 \\
  f(2.0001) &= 5.0001
\end{align*}
\]

Therefore

\[
\lim_{x \to 2} x + 3 = 5
\]

**Example 1.12.**

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1}
\]

**Solution.** At \( x_0 = 1 \) the function is undefined (denominator is zero), so by direct substitution we obtain the form \( 0/0 \). The form \( 0/0 \) is called an indeterminate. Therefore

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} x + 1 = 2
\]

**Remark 1.5.** We can also find

\[
\lim_{x \to x_0} f(x) = \pm \infty
\]

when the values of function increases(decreases) without bound. Of course we can also find

\[
\lim_{x \to \pm \infty} f(x) = g
\]
when the values of function approach $g$ as $x$ increases (decreases) without bound. Therefore we can find the following limits

$$\lim_{x \to -\infty} f(x) = -\infty \quad \lim_{x \to -\infty} f(x) = g \quad \lim_{x \to -\infty} f(x) = \infty$$

$$\lim_{x \to x_0} f(x) = -\infty \quad \lim_{x \to x_0} f(x) = g \quad \lim_{x \to x_0} f(x) = \infty$$

$$\lim_{x \to x_0} f(x) = -\infty \quad \lim_{x \to x_0} f(x) = g \quad \lim_{x \to x_0} f(x) = \infty$$

Let us formulate some obvious theorems

**Theorem 1.1.** It there exist limits $\lim_{x \to x_0} f(x)$, $\lim_{x \to x_0} g(x)$, then

1. $\lim_{x \to x_0} (c \cdot f(x)) = c \cdot \lim_{x \to x_0} f(x)$, for $c \in \mathbb{R}$,
2. $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$,
3. $\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)$,
4. $\lim_{x \to x_0} (f(x) \cdot g(x)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$,
5. $\lim_{x \to x_0} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$ if $\lim_{x \to x_0} f(x) \neq 0$,
6. $\lim_{x \to x_0} (f(x))^{g(x)} = \left( \lim_{x \to x_0} f(x) \right)^{\lim_{x \to x_0} g(x)}$.

**Theorem 1.2.**

1. $\lim_{x \to x_0} c = c$, for $c \in \mathbb{R}$, (limits of constant function is constant)
2. $\lim_{x \to x_0} x = x_0$
3. $\lim_{x \to \pm \infty} \frac{1}{x^k} = 0$, for $k > 0$
4. $\lim_{x \to \infty} x^k = \infty$, for $k > 0$
5. $\lim_{x \to -\infty} x^k = \begin{cases} \infty & \text{when } k \text{ is even} \\ -\infty & \text{when } k \text{ is odd} \end{cases}$

**Example 1.13.** Find

1. $\lim_{x \to -2} \frac{x^2 - 6}{x + 2} = \frac{2^2 - 6}{2 + 2} = \frac{-2}{4} = -\frac{1}{2}$
2. $\lim_{x \to \infty} x^2 = \infty$

3. $\lim_{x \to -1} \frac{x^2 - 2}{x^2 + 1} = \lim_{x \to -1} \frac{(x+1)x}{(x^2-x+1)} = \lim_{x \to -1} \frac{x}{x^2 - 1} = -\frac{1}{3} = -\frac{1}{3}$

4. $\lim_{x \to -\infty} \frac{x^3}{2x^2 - 7} = \lim_{x \to -\infty} \frac{x}{2 - \frac{7}{x}} = -\infty$

5. $\lim_{x \to \infty} \frac{x^2 + 3x}{2x^2 - 7} = \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{7}{x^2}}{2 - \frac{7}{x}} = \frac{1}{2}$

6. $\lim_{x \to \infty} \left(\frac{2}{x+1}\right)^{\frac{x+1}{x}} = \left(\lim_{x \to \infty} \left(\frac{2}{1+\frac{1}{x}}\right)^{\frac{3+\frac{1}{x}}{x}}\right) = 2^3 = 8$

7. $\lim_{x \to -\infty} \left(x^7 + 2x^4 + x^3 - 21x + 2\right) = \lim_{x \to -\infty} (x^7(1 + \frac{2}{x^2} + \frac{1}{x^3} - \frac{21}{x^6} + \frac{2}{x^7})) = -\infty$

**Theorem 1.3** (Some important limits).

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{\sin kx}{kx} = 1$

3. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

4. $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^\frac{x}{a} = e$

5. $\lim_{x \to \pm\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

**Example 1.14.**

1. $\lim_{x \to 0} \frac{\sin 7x - \sin 5x}{x} = \lim_{x \to 0} \left(\frac{7\sin 7x}{7x} - \frac{5\sin 5x}{5x}\right) = 7 \lim_{x \to 0} \frac{\sin 7x}{7x} - 5 \lim_{x \to 0} \frac{\sin 5x}{5x} = 7 - 5 = 2$

2. $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x = e^4$

3. $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^{\sqrt{x}} = \lim_{x \to \infty} \left((1 + \frac{4}{x})^{\sqrt{x}}\right) = e^{\lim_{x \to \infty} \frac{4}{x}} = e^0$

### 1.3.1 One-sides limits

When considering a limit, we need to consider what happens just to the left and right of the point in questions.
Definition 1.8. If $f(x)$ approaches $g$ as $x$ tends to $x_0$ from the left ($x < x_0$) we write

$$\lim_{x \to x_0^-} f(x) = g$$

and as $x$ tends to $x_0$ from the right ($x > x_0$) we write

$$\lim_{x \to x_0^+} f(x) = g$$

So we can consider the following limits

$$\lim_{x \to x_0^-} f(x) = -\infty \quad \lim_{x \to x_0^-} f(x) = g \quad \lim_{x \to x_0^+} f(x) = g$$

Theorem 1.4. Function $f$ has a limit $g$ in $x_0$, if and only if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = g$$

Example 1.15. Find $\lim_{x \to -1} f(x)$ where

$$f(x) = \begin{cases} x^3 + 4 & \text{for } x \leq -1 \\ x + 4 & \text{for } x > -1 \end{cases}$$
Solution.

\[ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x^3 + 4) = 3 \]

and

\[ \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x + 4) = 3 \]

due to the function \( f \) has a limit at \( x_0 = -1 \).

**Example 1.16.** Find

\[ \lim_{x \to 3} \frac{x^2 + x - 12}{(x - 3)^2} \]

**Solution.** We have

\[ \lim_{x \to 3} \frac{x^2 + x - 12}{(x - 3)^2} = \lim_{x \to 3} \frac{(x - 3)(x + 4)}{(x - 3)(x - 3)} = \lim_{x \to 3} \frac{x + 4}{x - 3} \]

Limit does not exist because the one side limits are different.

\[ \lim_{x \to 3^+} \frac{x + 4}{x - 3} = \frac{7}{0^+} = \infty \]

\[ \lim_{x \to 3^-} \frac{x + 4}{x - 3} = \frac{7}{0^-} = -\infty \]

where \( 0^+ \) denotes small positive numbers and \( 0^- \) denotes small negative numbers.

**1.3.2 Indeterminate forms**

The indeterminate forms are expressions for which we have no clear reply.

\[ \frac{0}{0}, \quad \infty \cdot \infty, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0, \quad 0^0 \]

**Example 1.17.**

1. let \( f(x) = x^2 \) and \( g(x) = x^3 \), so

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \left[ \frac{\infty}{\infty} \right] = \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0 \]

2. let \( f(x) = x^4 \) and \( g(x) = x^3 \), so

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \left[ \frac{\infty}{\infty} \right] = \lim_{x \to \infty} \frac{x^4}{x^3} = \lim_{x \to \infty} x = \infty \]

3. let \( f(x) = 5x^2 \) and \( g(x) = 15x^2 \), so

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \left[ \frac{\infty}{\infty} \right] = \lim_{x \to \infty} \frac{5x^2}{15x^2} = \lim_{x \to \infty} \frac{1}{3} = \frac{1}{3} \]
1.4 Continuity

Definition 1.9. A function $f$ is continuous at $x_0$ if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

Example 1.18. The function

$$f(x) = \begin{cases} x^3 + 4 & \text{for } x \leq -1 \\ x + 4 & \text{for } x > -1 \end{cases}$$

is continuous at $x_0 = -1$, because $f(-1) = \lim_{x \to -1} f(x) = 3$.

Example 1.19. The function

$$f(x) = \begin{cases} x^2 + 3 & \text{for } x < 0 \\ 2 - x^2 & \text{for } x \geq 0 \end{cases}$$

is not continuous at $x_0 = 0$, because limit in this point does not exist

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 3) = 3, \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x^2) = 2.$$
Theorem 1.5. The sum, difference, product and quotient \( \frac{f(x)}{g(x)} \) in case of \( g(x) \neq 0 \) of continuous functions is continuous.

Theorem 1.6. All of elementary functions are continuous.

Definition 1.10. The function \( f \) is continuous if it is continuous at every point of its domain \( x \in D_f \).
1.5 Review Exercises

Exercise 1.1. For the function

\[ f(x) = \begin{cases} 
3^x & \text{for } |x| \leq 2 \\
3x^2 - 3 & \text{for } |x| > 2 
\end{cases} \]

compute: \( f(-1), f(0), f(2), f(-8), f(8) \).

Exercise 1.2. Find the domain of functions:

\begin{itemize}
  \item[a)] \( f(x) = \frac{x^2}{x+1} \)
  \item[b)] \( f(x) = \sqrt{1-x^2} \)
  \item[c)] \( f(x) = \frac{1}{\sqrt{x^2-4x}} \)
  \item[d)] \( f(x) = (x-2)\sqrt{\frac{1+x}{1-x}} \)
  \item[e)] \( f(x) = \sqrt{2+x-x^2} + \frac{1}{\sqrt{x^2-3x}} \)
  \item[f)] \( f(x) = \frac{1}{x^2-x-2} \)
  \item[g)] \( f(x) = \frac{x^3-1}{x-1} \)
  \item[h)] \( f(x) = \frac{1}{\sqrt{x^2-1}} \)
\end{itemize}

Exercise 1.3. Find the composite function \( f(f(x)), f(g(x)), g(f(x)), g(g(x)) \), if:

\begin{itemize}
  \item[a)] \( f(x) = x^2, g(x) = 2^x \)
  \item[b)] \( f(x) = 2 + \cos x, g(x) = \sqrt{x} \)
\end{itemize}

Exercise 1.4. What are the functions of composed function:

\begin{itemize}
  \item[a)] \( f(x) = (1-3x)^5 \)
  \item[b)] \( f(x) = \frac{1}{(1-x^2)^4} \)
  \item[c)] \( f(x) = \sqrt[3]{(4+3x)^2} \)
\end{itemize}

Exercise 1.5. Find limits:

\begin{itemize}
  \item[a)] \( \lim_{x \to 1} \left( x^2 + 5x - 6 \right) \)
  \item[b)] \( \lim_{x \to -2} \frac{x^2+1}{x^2 - 1} \)
  \item[c)] \( \lim_{x \to 2} x\sqrt{x^2 + 5} \)
  \item[d)] \( \lim_{x \to 0} \frac{x^2 \cos x}{3x} \)
\end{itemize}

Exercise 1.6. Find limits:

\begin{itemize}
  \item[a)] \( \lim_{x \to -\infty} \frac{x^2 + 1}{x^5 + x} \)
  \item[b)] \( \lim_{x \to -\infty} \frac{2x^2 + 3x - 7}{x^2 + 4x - 2} \)
  \item[c)] \( \lim_{x \to -\infty} \frac{x^3 - 2x^2}{5x^3 + x^2 - x + 2} \)
  \item[d)] \( \lim_{x \to -\infty} \frac{x + 1}{x^2 - 1} \)
  \item[e)] \( \lim_{x \to -\infty} \frac{x^3 + 5x}{x - 1} \)
  \item[f)] \( \lim_{x \to -\infty} \frac{x^2 - 1}{x + 1} \)
  \item[g)] \( \lim_{x \to -\infty} (x^4 + 2x^2 + 3) \)
  \item[h)] \( \lim_{x \to -\infty} (-2x^3 + 5x - 7) \)
  \item[i)] \( \lim_{x \to -\infty} (x^4 + 5x - 6) \)
  \item[j)] \( \lim_{x \to -\infty} (-2x^6 + 5x - 4) \)
  \item[k)] \( \lim_{x \to -\infty} (x^3 + 2x - 7) \)
  \item[l)] \( \lim_{x \to -\infty} (-2x^5 + 6x^4 - 3x + 7) \)
\end{itemize}

Exercise 1.7. Find limits:

\begin{itemize}
  \item[a)] \( \lim_{x \to 2} \frac{x^2 + 3x - 16}{x - 2} \)
  \item[b)] \( \lim_{x \to -1} \frac{x^2 - x - 2}{x + 1} \)
  \item[c)] \( \lim_{x \to -3} \frac{x^2 - 2x - 3}{3 - x} \)
  \item[d)] \( \lim_{x \to -4} \frac{x^2 + 3x - 4}{x^2 + 5x + 4} \)
\end{itemize}
Exercise 1.8. Find one-side limits of function $f$ at $x_0$, if:

a) $f(x) = \frac{1}{x - 3}, \; x_0 = 3$  
b) $f(x) = \frac{x + 1}{x - 1}, \; x_0 = 1$  
c) $f(x) = \frac{1}{(3 - x)^2}, \; x_0 = 3$

d) $f(x) = \frac{1}{x^2 - 4}, \; x_0 = 2$  
e) $f(x) = \frac{x + 3}{x - 1}, \; x_0 = 1$  
f) $f(x) = \frac{4}{x^2 - 4}, \; x_0 = 2$

g) $f(x) = e^{\frac{1}{x}}$, $x_0 = -2$

Exercise 1.9. Find limits:

a) $\lim_{x \to -2} \frac{x + 1}{x - 2}$  
b) $\lim_{x \to 1} \frac{x^2 + 4x + 3}{x - 1}$  
c) $\lim_{x \to -2} \frac{x^2 + 2x - 3}{x + 2}$  
d) $\lim_{x \to -3} \frac{x^2 - 5x}{x + 3}$

Exercise 1.10. Find limits:

a) $\lim_{x \to 0} \frac{\sin 2x}{3x}$  
b) $\lim_{x \to 0} \frac{\sin 5x}{3x}$  
c) $\lim_{x \to 0} x \cot x$

Exercise 1.11. Find limits:

a) $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x$  
b) $\lim_{x \to \infty} \left(\frac{x + 1}{x - 2}\right)^x$  
c) $\lim_{x \to \infty} \left(\frac{x^2 + 3}{x^2 + 7}\right)^x$  
d) $\lim_{x \to \infty} \left(\frac{3x - 4}{3x + 2}\right)^{x+1}$  
e) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x+3}$  
f) $\lim_{x \to \infty} \left(1 - \frac{3}{x}\right)^{2x-1}$

Exercise 1.12. Discuss the continuity of functions $f$

a) $f(x) = \begin{cases} 
\frac{x^2 - 3}{x + 1} & \text{for } x \geq 0 \\
\frac{x - 3}{x + 1} & \text{for } x < 0 
\end{cases}$  
b) $f(x) = \begin{cases} 
\frac{x + 5}{6 - x} & \text{for } x \geq 0 \\
\frac{2x}{6 - x} & \text{for } x < 0 
\end{cases}$

c) $f(x) = \begin{cases} 
x + 5 & \text{for } x < 0 \\
2x^2 - 4 & \text{for } x \geq 0 
\end{cases}$  
d) $f(x) = \begin{cases} 
x + 1 & \text{for } x \geq 2 \\
\frac{x - 1}{x} & \text{for } x < 2 
\end{cases}$

e) $f(x) = \begin{cases} 
x + 1 & \text{for } x \neq 1 \\
0 & \text{for } x = 1 
\end{cases}$  
f) $f(x) = \begin{cases} 
\frac{3x}{2 - x} & \text{for } x > 2 \\
x & \text{for } x \leq 2 
\end{cases}$

Answers

Exercise 1.1. $\frac{13}{3}, 1, 9, 61, 61.$

Exercise 1.2. a) $x \in \mathbb{R} \setminus \{-1\}$, b) $x \in [-1, 1]$, c) $x \in (-\infty, 0] \cup [4, \infty)$, d) $x \in [-1, 1)$, e) $x \in [-1, 0)$, f) $x \in \mathbb{R} \setminus \{-1, 2\}$, g) $x \in \mathbb{R} \setminus \{1\}$, h) $x \in (1, \infty) \cup (-\infty, -1)$, i) $x \in (-\frac{1}{3}, 1]$, j) $x \in \mathbb{R}$, t) $x \in [-1, 1] \setminus \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

Exercise 1.3. a) $x^4, 2^{2x}, 2^{x^2}, 2^{2^x}$, b) $\cos(\cos(x + 2)) + 2$, $\cos\left(\sqrt[3]{x} + 2\right)$, $\sqrt[3]{\cos(x + 2)}$, $\sqrt{x}$. 

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Exercise 1.4. a) $f(x) = z(w(x))$, where $w(x) = 1 - 3x^2$, $z(x) = x^5$, b) $f(x) = z(s(w(x)))$, where $w(x) = 1 - x^2$, $s(x) = x^4$, $z(x) = \frac{1}{x}$, c) $f(x) = z(s(w(x)))$, where $w(x) = 4 + 3x$, $s(x) = x^2$, $z(x) = \sqrt[3]{x}$.

Exercise 1.5. a) 0, b) $\frac{5}{2}$, c) 6, d) 0.

Exercise 1.6. a) 0, b) 2, c) $\frac{1}{3}$, d) 0, e) $\infty$, f) $-\infty$, g) $\infty$, h) $-\infty$, i) $\infty$, j) $-\infty$, k) $-\infty$, l) $\infty$.

Exercise 1.7. a) $\infty$, $-\infty$, b) $-3$, $-3$, c) $-4$, $-4$, d) $\frac{5}{3}$.

Exercise 1.8. a) $\infty$, $-\infty$, b) $-\infty$, $\infty$, c) $\infty$, $\infty$, d) $-\infty$, $\infty$, e) 0, $\infty$, f) 0, $\infty$, h) 0, $\infty$.

Exercise 1.9. a) $\infty$, b) $-\infty$, $\infty$, c) $\infty$, $-\infty$, d) $-1$.

Exercise 1.10. a) $\frac{2}{3}$, b) $\frac{5}{3}$, c) 0.

Exercise 1.11. a) $e^4$, b) $e^3$, c) $e^{-4}$, d) $e^{-\frac{2}{3}}$, e) $e$, f) $e^{-6}$.

Exercise 1.12. a) continuous, b) continuous, c) discontinuous for $x_0 = 0$, d) discontinuous for $x_0 = 2$, e) discontinuous for $x_0 = 1$, f) discontinuous for $x_0 = 2$.
Chapter 2

Derivatives of functions

2.1 The derivative of function

Definition 2.1. The derivative of function $f$ at $x_0$ we call the limit (if it exists)

$$
\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}
$$

and denote by $f'(x_0)$.

If the function $f$ has a derivative at $x_0$, then we say that $f$ is differentiable at $x_0$.

![Diagram of derivatives](image)

Definition 2.2. The expression

$$
\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}
$$

we called a difference quotient for the function $f$ for $x_0 \in D_f$. 

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We usually denote

- \( \Delta f = f(x_0 + \Delta x) - f(x_0) \)
- \( f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \)
- The other symbols used for derivative
  
  \[
  y' \quad f'(x) \quad \frac{du}{dx} \\
  \frac{df}{dx}(x) \quad \frac{d}{dx}f(x) \quad \frac{\partial}{\partial x}f(x)
  \]

**Remark 2.1.** The derivative \( f' \) expresses the growth rate function \( f \).

### 2.1.1 Rules of derivatives

In this section we present some simple rules for producing derivatives, without resorting to the definition.

**Theorem 2.1.** If functions \( f \) and \( g \) are differentiable at \( x_0 \), and \( c \in \mathbb{R} \), then

1. \( (c \cdot f(x_0))' = c \cdot f'(x_0) \)
2. \( (f(x_0) \pm g(x_0))' = f'(x_0) \pm g'(x_0) \)
3. \( (f(x_0) \cdot g(x_0))' = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0) \)
4. \( \left( \frac{f(x_0)}{g(x_0)} \right)' = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)} \) (for \( g(x_0) \neq 0 \))

Derivatives of some functions

\[
\begin{align*}
(c)' &= 0 \quad & (x^p)' &= px^{p-1} \\
(sin \ x)' &= \cos \ x \quad & (cos \ x)' &= -\sin \ x \\
(e^x)' &= e^x \quad & (a^x)' &= a^x \ln a \\
(ln \ x)' &= \frac{1}{x} \quad & (\log_a \ x)' &= \frac{1}{x \ln a}
\end{align*}
\]
Example 2.1. Find the derivative of functions:

1. \( f(x) = x^2 + 3x - 3 \)
2. \( f(x) = x^{-4} - x^{12} - \sin x \)
3. \( f(x) = e^x + \ln x + 7 \)
4. \( f(x) = x^3 \cos x \)
5. \( f(x) = \frac{\sqrt{x-1}}{x^2 + x + 7} \)

Solution.

1. \( f'(x) = (x^2 + 3x - 3)' = (x^2)' + (3x)' - (3)' = 2x + 3 - 0 = 2x + 3 \)
2. \( f'(x) = (x^{-4} - x^{12} - \sin x)' = (x^{-4})' - (x^{12})' - (\sin x) = -4x^{-5} - 12x^{11} - \cos x \)
3. \( f'(x) = (e^x + \ln x + 7)' = (e^x) + (\ln x)' + (7)' = e^x + \frac{1}{x} \)
4. \( f'(x) = (x^3 \cos x)' = (x^3)' \cos x + x^3(\cos x)' = 3x^2 \cos x + x^3(- \sin x) = 3x^2 \cos x - x^3 \sin x \)
5. 
\[
    f'(x) = \left( \frac{x - 1}{x^2 + x + 7} \right)' = \frac{(x - 1)'(x^2 + x + 7) - (x - 1)(x^2 + x + 7)'}{(x^2 + x + 7)^2} = \\
    = \frac{1 \cdot (x^2 + x + 7) - (x - 1)(2x + 1)}{(x^2 + x + 7)^2} = \\
    = \frac{x^2 + x + 7 - (2x^2 - x - 1)}{(x^2 + x + 7)^2} = \\
    = \frac{-x^2 + 2x + 8}{(x^2 + x + 7)^2}
\]

2.1.2 Graphical interpretation of derivative

Geometrically derivative of the function \( y = f(x) \) at that point \( x_0 \) is equal to the slope of tangent line to the graph of the function at that point.
Chapter 2. Derivatives of functions

The tangent line to the graph of $f$ at $P(x_0, f(x_0))$ is of the form

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Let us notice, that $\tan \alpha = f'(x_0)$.

**Example 2.2.** Find the tangent line to $y = x^2 + 1$ at $P(2, 5)$.

**Solution.** We have $f(x) = x^2 + 1$ and $x_0 = 2$, so $f'(x) = 2x$ and $f'(2) = 4$. Therefore

$$y = f'(x_0)(x - x_0) + f(x_0) = 4(x - 2) + 5 = 4x - 3$$
Definition 2.3. If function $g$ is a function that is differentiable at a point $x_0$ (i.e. the derivative $g'(x_0)$ exists) and $f$ is a function that is differentiable at $g(x_0)$, then the composite function $f(g(x))$ is differentiable at $x_0$, and the derivative is of the form

$$(f(g(x_0)))' = f'(g(x_0)) \cdot g'(x_0).$$

Example 2.3. Find the derivative of functions

1. $h(x) = (3x^2 - 6x + 5)^3$

2. $h(x) = \ln \sin x$.

3. $h(x) = \cos 7x$

Solution.

1. $h(x) = (3x^2 - 6x + 5)^3$. Let $h = f(g(x))$, where $f(x) = x^3$ and $g(x) = 3x^2 - 6x + 5$. Then $f'(x) = 3x^2$ and $g'(x) = 6x - 6$. So derivative of composite function is of the form

$$h'(x) = 3(3x^2 - 6x + 5)^2(6x - 6)$$
2. \( h(x) = \ln \sin x \). Let \( h = f(g(x)) \), where \( f(x) = \ln x \) and \( g(x) = \sin x \). Then \( f'(x) = \frac{1}{x} \) and \( g'(x) = \cos x \). So derivative of composite function is of the form

\[
h'(x) = \frac{1}{\sin x} \cos x = \frac{\cos x}{\sin x} = \cot x
\]

3. \( h(x) = \cos 7x \). Let \( h = f(g(x)) \), where \( f(x) = \cos x \) and \( g(x) = 7x \). Then \( f'(x) = -\sin x \), and \( g'(x) = 7 \). So derivative of composite function is of the form

\[
h'(x) = (\sin 7x) \cdot 7 = -7 \sin 7x
\]

### 2.2 Properties of derivatives

#### 2.2.1 L’Hospital rule

L’Hospitala rule enables to calculate the limits of the function in case of indeterminate forms, such as \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \).

**Theorem 2.2.** If

1. \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \) or \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \infty \),

2. \( \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \),

then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.
\]

**Example 2.4.** Find limits:

1. \[
\lim_{x \to 1} \frac{x^2 - 1}{\ln x} = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \Rightarrow \lim_{x \to 1} \frac{2x}{\frac{1}{x}} = \lim_{x \to 1} 2x^2 = 2
\]

2. \[
\lim_{x \to \infty} \frac{e^{x^2}}{2 - 3x} = \left[ \frac{\infty}{-\infty} \right] \Rightarrow \lim_{x \to \infty} \frac{e^{x^2} 2x}{-3} = -\infty
\]

3. \[
\lim_{x \to \infty} \frac{x}{e^x} = \left[ \frac{\infty}{\infty} \right] \Rightarrow \lim_{x \to \infty} \frac{1}{e^x} = 0
\]

---

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4. 
\[
\lim_{x \to 0^+} \frac{\sqrt{x}}{\ln(1 - x)} = \left[ \frac{0}{0} \right] = \frac{1/2}{0^+} = -\infty \\
\lim_{x \to 0^+} \frac{x - 1}{2\sqrt{x}} = \left[ -1 \right] = -\infty
\]

5. 
\[
\lim_{x \to \infty} \frac{x}{\ln^2 x} = \left[ \frac{\infty}{\infty} \right] = \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{1}{2\ln x} = \lim_{x \to \infty} \frac{x}{2\ln x} = \left[ \frac{\infty}{\infty} \right] \\
\lim_{x \to \infty} \frac{1}{2\sqrt{x}} = \lim_{x \to \infty} \frac{x}{2} = \infty
\]

2.2.2 Monotonicity of functions

**Theorem 2.3.** If \( f \) is differentiable on the interval \((a, b)\) then

- if \( f'(x) > 0 \), then \( f \) is increasing on \((a, b)\),

- if \( f'(x) < 0 \), then \( f \) is decreasing on \((a, b)\).

**Example 2.5.** Determine the intervals of monotonicity of the function \( f(x) = x^2 \).

**Solution.** If \( f(x) = x^2 \), then \( f'(x) = 2x \). And

- \( f'(x) > 0 \iff 2x > 0 \iff x > 0 \)

- \( f'(x) < 0 \iff 2x < 0 \iff x < 0 \)

Therefore the function \( f(x) = x^2 \) is increasing on the interval \((0, \infty)\) and decreasing on the interval \((-\infty, 0)\).

**Example 2.6.** Determine the intervals of monotonicity of the function \( f(x) = \frac{x}{1+x^2} \).

**Solution.** If \( f(x) = \frac{x}{1+x^2} \), then \( f'(x) = \frac{1-x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \). And

- \( f'(x) > 0 \iff \frac{1-x^2}{(1+x^2)^2} > 0 \iff 1 - x^2 > 0 \iff x \in (-1, 1) \)

- \( f'(x) < 0 \iff \frac{1-x^2}{(1+x^2)^2} < 0 \iff 1 - x^2 < 0 \iff x \in (-\infty, -1) \cup (1, \infty) \)
Therefore the function $f(x) = \frac{x}{1+x^2}$ is increasing on the interval $(-1, 1)$ and decreasing on the interval $x \in (-\infty, -1)$ and on the interval $x \in (1, \infty)$.

### 2.2.3 Local extrema

**Definition 2.4.** The function $f : (a, b) \to \mathbb{R}$ has a **local minimum** at point $x_0 \in (a, b)$, if

$$f(x_0) \leq f(x)$$

for all $x \in (a, b)$.

**Definition 2.5.** The function $f : (a, b) \to \mathbb{R}$ has a **local maximum** at point $x_0 \in (a, b)$, if

$$f(x_0) \geq f(x)$$

for all $x \in (a, b)$.
Minimum and maximum of the function we usually call extremum of the function.

**Definition 2.6.** The point \( x_0 \) with \( f'(x_0) = 0 \) is called the critical point of the function \( f \).

**Theorem 2.4.** If \( f \) is differentiable at \( x_0 \in (a, b) \) and has at this point extremum, then
\[
f'(x_0) = 0.
\]

**Remark 2.2.** The converse theorem is false.

**Example 2.7.** For instance, the derivative of function \( f(x) = x^3 \) is of the form \( f'(x) = 3x^2 \) and \( f'(0) = 0 \) but \( f \) has not an extremum at 0, because
\[
f'(x) > 0 \Leftrightarrow 3x^2 > 0 \Leftrightarrow x \in (-\infty, 0) \cup (0, \infty)
\]

and

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<th>( D_f )</th>
<th>(-\infty, 0)</th>
<th>0</th>
<th>((0, \infty))</th>
</tr>
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<tbody>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>↗</td>
<td>↗</td>
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</table>
Therefore we can formulate the following theorem

**Theorem 2.5.** Let \( f : (a,b) \to \mathbb{R} \) is differentiable and \( x_0 \in (a,b) \). If \( f'(x_0) = 0 \) and \( f' \) changes the sign when passing through the point \( x_0 \), then the function \( f \) has a local extremum at the point \( x_0 \). Moreover if

\[
\begin{cases}
  f'(x) < 0, & \text{to the left of } x_0 \\
  f'(x) > 0, & \text{to the right of } x_0
\end{cases}
\]

then \( f \) has a local minimum; or if

\[
\begin{cases}
  f'(x) > 0, & \text{to the left of } x_0 \\
  f'(x) < 0, & \text{to the right of } x_0
\end{cases}
\]

then \( f \) has a local maximum;

**Example 2.8.** Determine the extrema of function \( f(x) = x^3e^x + 1 \).

**Solution.** Step by step we can find extremum of function.

- domain: \( D_f = \mathbb{R} \)
- derivative: \( f'(x) = 3x^2e^x + x^3e^x = x^2e^x(3 + x) \)
- critical points: \( f'(x) = 0 \iff x = 0 \text{ or } x = -3 \)
• sign of the derivative: because of $x^2 \geq 0$ and $e^x > 0$ we have

$$f'(x) > 0 \iff x + 3 > 0 \iff x > -3$$

$$f'(x) < 0 \iff x < -3 \text{ and } x \neq 0$$

So summing up

<table>
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<tr>
<th>$D_f$</th>
<th>$(-\infty, -3)$</th>
<th>$-3$</th>
<th>$(-3, 0)$</th>
<th>$0$</th>
<th>$(0, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>$\downarrow$</td>
<td>min</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
</tr>
</tbody>
</table>

This function has no maximum, but a minimum for $x = -3$, $f_{min} = 1 - \frac{27}{e^3}$. At the point $x = 0$ the sign of derivative does not change, and therefore there is no extreme.

The graf of function $f(x) = x^3e^x + 1$.

2.2.4 The smallest and the largest value of the function over the interval

To determine the minimum and maximum value of the function over the interval, we should perform the following steps
Chapter 2. Derivatives of functions

1. determine the local extrema of functions on this interval,

2. find the extremal values

3. calculate values of the function at the ends of the interval

The smallest and largest of these numbers, are solutions for our problem.

Example 2.9. Find the smallest and the largest value of the function \( f(x) = x^4 - 4x \) over the interval \([-1, 2]\).

Solution. Derivative of this function is of the form \( f'(x) = 4x^3 - 4 \), so \( f'(x) = 0 \Leftrightarrow x = 1 \), then

\[
\begin{align*}
    f'(x) > 0 & \Leftrightarrow x > 1 \\
    f'(x) < 0 & \Leftrightarrow x < 1
\end{align*}
\]

For \( x = 1 \) function has minimum \( f_{\min}(1) = -3 \). Values of the function at the ends of the intervals are the following \( f(-1) = 5 \) end \( f(2) = 8 \). The smallest value of the given function over the interval \([-1, 2]\) are \(-3\) and the largest \(8\).
2.3 Second Derivative

Definition 2.7. The derivative of the derivative is called the second derivative. The second derivative is denoted by \( f'' \). So

\[
f''(x) = (f'(x))'.
\]

Example 2.10. Find the second derivative of the function \( f(x) = \sqrt[3]{(x - 2)^5} = (x - 2)^{\frac{5}{3}} \)

Solution.

\[
f'(x) = \frac{5}{3}(x - 2)^{\frac{2}{3}} \cdot 1 = \frac{5}{3}(x - 2)^{\frac{2}{3}}
\]

\[
f''(x) = \frac{5 \cdot 2}{3^3} (x - 2)^{-\frac{1}{3}} = \frac{10}{9} (x - 2)^{-\frac{1}{3}} = \frac{10}{\sqrt[3]{x - 2}}
\]

2.3.1 The second derivative test for the extremum

Theorem 2.6. Let \( f \) be a differentiable function with \( f'(x_0) = 0 \), and

1. if \( f''(x_0) > 0 \), then \( f \) has at the point \( x_0 \) minimum,

2. if \( f''(x_0) < 0 \), then \( f \) has at the point \( x_0 \) maximum.

Remark 2.3. In the case of \( f''(x_0) = 0 \) we must test the first derivative of \( f \).

Example 2.11. Find the extrema of the function \( f(x) = \frac{2x}{x^2 + 1} \).

Solution. The first derivative of the function \( f \) is of the form

\[
f'(x) = \frac{2(x^2 + 1) - 2x \cdot 2x}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}
\]

We determine the critical points.

\[
f'(x) = 0 \Leftrightarrow 2 - 2x^2 = 0 \Leftrightarrow (1 - x)(1 + x) = 0 \Leftrightarrow x = -1 \text{ or } x = 1
\]

We calculate the second derivative

\[
f''(x) = \frac{-4x(x^2 + 1)^2 - (2 - 2x^2)2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{-4x(x^2 + 1)(x^2 + 1 + 2 - 2x^2)}{(x^2 + 1)^4}
\]

\[
= \frac{-4x(x^2 + 1)(3 - x^2)}{(x^2 + 1)^4}
\]

because of \( f''(-1) > 0 \) and \( f''(1) < 0 \) we find, that \( f \) has minimum at the point \( x = -1 \) and maximum at \( x = 1 \). Moreover \( f_{\text{min}}(-1) = -1 \) and \( f_{\text{max}}(1) = 1 \).
2.3.2 Convex and concave function

Convex function

**Definition 2.8.** The function \( f \) is convex over the interval \((a, b)\) if for any \( x, y \in (a, b) \) such that \( x \neq y \) and any \( t \in (0, 1) \)

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

It means, that the graf of the function is \( \cup \)-shaped.

![Convex function graph](image)

**Theorem 2.7.** In any interval on which \( f''(x) > 0 \) the function is convex.

Concave function

**Definition 2.9.** The function \( f \) is concave over the interval \((a, b)\) if for any \( x, y \in (a, b) \) such that \( x \neq y \) and any \( t \in (0, 1) \)

\[
f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)
\]

It means, that the graf of the function is \( \cap \)-shaped.
Theorem 2.8. In any interval on which $f''(x) < 0$ the function is concave.

Example 2.12. Determine intervals of convexity for the function $f(x) = x^4 - 6x^2 + 1$.

Solution.

- the first derivative $f'(x) = 4x^3 - 12x$

- the second derivative $f''(x) = 12x^2 - 12$

- the sign of the second derivative

$$f''(x) > 0 \iff 12x^2 - 12 > 0 \iff (x - 1)(x + 1) > 0 \iff x \in (-\infty, -1) \cup (1, \infty)$$

$$f''(x) < 0 \iff x \in (-1, 1)$$
The function is convex for intervals \((-\infty, -1)\) and \((1, \infty)\), concave for \((-1, 1)\).

### 2.3.3 Points of inflection

**Definition 2.10.** Any point on the graph of a function at which the convexity changes is called an inflection point.

**Theorem 2.9.** If the function \(f\) is twice differentiable at the point \(x_0 \in (a, b)\) and the point \((x_0, f(x_0))\) is an inflection point of the graph \(f\), then

\[
f''(x) = 0
\]

**Example 2.13.** Find the inflection points of the function \(f(x) = e^{-x^2}\).

**Solution.**

- the first derivative \(f'(x) = -2xe^{-x^2}\)
- the second derivative \(f''(x) = -2e^{-x^2} - 2xe^{-x^2}(-2x) = 2e^{-x^2}(2x^2 - 1)\)
- the zeros of the second derivative \(f''(x) = 0 \iff 2x^2 - 1 = 0 \iff x = \pm \frac{\sqrt{2}}{2}\)
• the sign of the second derivative

\[ f''(x) > 0 \iff 2x^2 - 1 > 0 \]
\[ \iff (x - \frac{\sqrt{2}}{2})(x + \frac{\sqrt{2}}{2}) > 0 \]
\[ \iff x \in (-\infty, -\frac{\sqrt{2}}{2}) \cup \left( \frac{\sqrt{2}}{2}, \infty \right) \]
\[ f''(x) < 0 \iff x \in \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \]

because of \( f(-\frac{\sqrt{2}}{2}) = \frac{1}{\sqrt{e}} \) and \( f(\frac{\sqrt{2}}{2}) = \frac{1}{\sqrt{e}} \), we determine, that points \( \left( -\frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}} \right) \) and \( \left( \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}} \right) \) are inflection points of the given function.

**Example 2.14.** Determine where the function \( f(x) = \frac{x^3}{2(4 - x^2)} \) is increasing and decreasing, and where graph is convex or concave. Find the limits of the function and points of inflection and sketch the graph.

**Solution.**

1. the domain of the function \( D_f = \mathbb{R} - \{-2, 2\} \).

2. limits of function

\[
\begin{align*}
\lim_{x \to -\infty} f(x) &= \infty \\
\lim_{x \to \infty} f(x) &= -\infty \\
\lim_{x \to -2^-} f(x) &= \left[ \frac{8}{0} \right] = \infty \\
\lim_{x \to -2^+} f(x) &= \left[ -\frac{8}{0} \right] = -\infty \\
\lim_{x \to 2^-} f(x) &= \left[ \frac{8}{0} \right] = \infty \\
\lim_{x \to 2^+} f(x) &= \left[ -\frac{8}{0} \right] = -\infty
\end{align*}
\]

3. the first derivative

\[
f'(x) = \frac{3x^2 \cdot 2(4 - x^2) - x^3(-4x)}{4(4 - x^2)^2} = \frac{6x^2(4 - x^2) + 4x^4}{4(4 - x^2)^2} = \frac{2x^2(12 - 3x^2 + 2x^2)}{4(4 - x^2)^2} = \frac{x^2(12 - x^2)}{2(4 - x^2)^2}
\]

then

\[
f'(x) = 0 \iff x = 0 \text{ lub } x = -2\sqrt{3} \text{ lub } x = 2\sqrt{3}
\]
\[f'(x) < 0 \iff x \in (-\infty, -2\sqrt{3}) \cup (2\sqrt{3}, \infty)\]
\[f'(x) > 0 \iff x \in (-2\sqrt{3}, -2) \cup (-2, 0) \cup (0, 2) \cup (2, 2\sqrt{3})\]
4. the second derivative

\[
f''(x) = \frac{(24x - 4x^3)2(4 - x^2)^2 - x^2(12 - x^2)4(4 - x^2)(-2x)}{4(4 - x^2)^4} = \frac{8x(4 - x^2)(24 - 10x^2 + x^4 + 12x^2 - x^4)}{4(4 - x^2)^2} = \frac{2x(2x^2 + 24)}{(4 - x^2)^3} = \frac{4x(x^2 + 12)}{(4 - x^2)^3}
\]

then

\[
f''(x) = 0 \iff x = 0
\]

\[
f''(x) < 0 \iff x \in (-2, 0) \cup (2, \infty)
\]

\[
f''(x) > 0 \iff x \in (-\infty, -2) \cup (0, 2)
\]

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<thead>
<tr>
<th>$D_f$</th>
<th>$(-\infty, -2\sqrt{3})$</th>
<th>$-2\sqrt{3}$</th>
<th>$(-2\sqrt{3}, -2)$</th>
<th>$-2$</th>
<th>$(-2, 0)$</th>
<th>$0$</th>
<th>$(0, 2)$</th>
<th>$2$</th>
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<tr>
<td>$f'(x)$</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>x</td>
<td>+</td>
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<td>$f''(x)$</td>
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<th>$(2, 2\sqrt{3})$</th>
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<td>_</td>
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</table>
The graph of $f(x) = \frac{x^3}{2(4-x^2)}$. 
2.4 Review Exercises

Exercise 2.1. Find the derivative of the given function:

1) \( f(x) = 3 \) 
2) \( f(x) = \frac{1}{2}x - \frac{3}{4} \) 
3) \( f(x) = \pi x - 123 \)
4) \( f(x) = \frac{1}{2} - \sqrt{3} + \sqrt{2}x \) 
5) \( f(x) = 2x^2 + 3x - 4 \) 
6) \( f(x) = \frac{1}{2}x^2 - 2 + 5x \)
7) \( f(x) = \frac{3}{4}x^2 - 3x - 12 \) 
8) \( f(x) = x^3 + 2x^2 - 5x + 7 \) 
9) \( f(x) = 4x^3 - 12x - 12\frac{1}{2} \)
10) \( f(x) = \frac{2x - 7}{x + 2} \) 
11) \( f(x) = \frac{x - 1}{3 + x} \) 
12) \( f(x) = \frac{2 - x}{4x + 6} \)
13) \( f(x) = \frac{x^2 - 2x + 4}{x^2 + 3} \) 
14) \( f(x) = \frac{2x + 5}{3x^2 - 5x + 6} \) 
15) \( f(x) = \frac{x^2 - 3}{x + 1} \)
16) \( f(x) = x\sqrt{1 + x^2} \) 
17) \( f(x) = e^x \) 
18) \( f(x) = \frac{x}{e^x} \)
19) \( f(x) = 2\sqrt{x} - 3\ln x + 1 \) 
20) \( f(x) = x\ln x \) 
21) \( f(x) = \frac{\ln x}{1 + x^2} \)
22) \( f(x) = 4\ln x \) 
23) \( f(x) = \sin x + \cos x \) 
24) \( f(x) = x^3\sin x \)
25) \( f(x) = \sqrt{x}\cos x \) 
26) \( f(x) = \frac{\sin x}{x^4 + 4} \) 
27) \( f(x) = \frac{\sin x - \cos x}{\sin x + \cos x} \)
28) \( f(x) = \sin^2 x + \cos^2 x \) 
29) \( f(x) = x^{\frac{3}{2}} - 2x^{2013} \) 
30) \( f(x) = (x^{10} + 3x^{-10})(\sin x + e^x) \)
31) \( f(x) = \sqrt{\frac{1 - x}{1 + x}} \) 
32) \( f(x) = \left(\frac{\sin x}{1 + \cos x}\right)^3 \) 
33) \( f(x) = \cos^3 4x \)
34) \( f(x) = (2x^3 - 1)^5 \) 
35) \( f(x) = \left(\frac{1 + x^2}{1 + x}\right)^5 \) 
36) \( f(x) = x + \frac{1}{x} + e \)

Exercise 2.2. Find the intervals of increase and decrease for the function:

a) \( f(x) = 2x^3 - 9x^2 + 12x - 17 \) 
b) \( f(x) = 2x^3 + 5x^2 - 4x - 11 \) 
c) \( f(x) = \frac{3x^2}{x - 2} \)

\[ d) f(x) = x^3 - 3x^2 + 3x - 19 \] 
\[ e) f(x) = -x^3 + x^2 + 21x + 3 \] 
\[ f) f(x) = \frac{x^2 - 4}{x^2 + 4} \]

\[ g) f(x) = \frac{4x^2 - 3x - 1}{4x^2 + 1} \] 
\[ h) f(x) = \frac{x^2}{x - 1} \] 
\[ i) f(x) = 2x + \sin x \]

Exercise 2.3. Find the extrema of the function:

a) \( f(x) = x^3 + 3x^2 - 9x + 7 \) 
b) \( f(x) = \frac{1}{3}x^3 - 3x^2 + 5x + 9 \) 
c) \( f(x) = \frac{x^2 + 1}{x^2 + 4} \)

\[ d) f(x) = -x^3 + 9x^2 - 24x + 17 \] 
\[ e) f(x) = \frac{3x}{x^2 + x + 1} \] 
\[ f) f(x) = 3x^5 + 5x^3 - 11 \]

\[ g) f(x) = x^4 - 4x^3 + 4x^2 - 11 \] 
\[ h) f(x) = x + \sqrt{x} \] 
\[ i) f(x) = e^x + e^{-x} \]
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Exercise 2.4. Find the smallest and the largest values of the given function over the given interval.

a) \( f(x) = x^4 - 32x, \ x \in (-2, 3) \)

b) \( f(x) = x^3 - 3x^2 + 6x - 5, \ x \in (-1, 1) \)

c) \( f(x) = 3x - x^3, \ x \in (0, 4) \)

d) \( f(x) = x^4 - x^2, \ x \in (-1, 2) \)

Exercise 2.5. Determine intervals of convexity of the given function:

a) \( f(x) = x^4 - 6x^2 - 6x \)

b) \( f(x) = \frac{x^2 - 5x + 6}{x + 1} \)

c) \( f(x) = \frac{x^3}{(x - 1)^3} \)

d) \( f(x) = \frac{x^4}{12} - \frac{x^3}{3} + x^2 \)

e) \( f(x) = (1 + x^2) e^x \)

f) \( f(x) = x^4 (x - 2)^4 \)

Exercise 2.6. By de L’Hospital rule calculate the limit:

a) \( \lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} \)

b) \( \lim_{x \to 0^+} x \ln x \)

c) \( \lim_{x \to -\infty} \frac{e^x - 1}{x} \)

d) \( \lim_{x \to 0} \frac{e^x - x - 1}{x^2} \)

e) \( \lim_{x \to 0} \frac{\ln (1 + x)}{x} \)

f) \( \lim_{x \to e} \frac{\ln x - 1}{x - e} \)

g) \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \)

h) \( \lim_{x \to 0} \frac{\sin x}{x} \)

i) \( \lim_{x \to 0} \frac{\sin x}{x \cos x} \)

j) \( \lim_{x \to +\infty} \frac{e^x}{x} \)

k) \( \lim_{x \to +\infty} \frac{\ln x}{x} \)

l) \( \lim_{x \to +\infty} \frac{\ln x}{\sqrt{x}} \)
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Answers

Exercise 3.1. 1) \( f'(x) = 0, 2) \ f'(x) = \frac{1}{2}, 3) \ f'(x) = \pi, 4) f'(x) = \sqrt{2}, 5) f'(x) = 4x + 3, 6) f'(x) = x + 5, 7) f'(x) = \frac{3}{2}x - 3, 8) f'(x) = 4x + 3x^2 - 5, 9) f'(x) = 12x^2 - 12, 10) f'(x) = \frac{11}{(x+2)^2}, 11) f'(x) = \frac{4}{(x+3)^2}, 12) f'(x) = -\frac{7}{2(2x+3)^4}, 13) f'(x) = \frac{x^2-x-3}{2(x+3)^2}, 14) f'(x) = \frac{30x+6x^2-37}{(3x^2-5)^2}, 15) f'(x) = \frac{2x^2+x^3}{(x+1)^2}, 16) f'(x) = -\frac{2x^2\ln x-x^2-1}{x(x+1)^2}, 17) f'(x) = e^x, 18) f'(x) = \frac{1}{x}, 19) f'(x) = \frac{1}{\sqrt{x}} - \frac{3}{x}, 20) f'(x) = \ln x + 1, 21) f'(x) = \frac{1}{2\sqrt{x}} \cos x - \sqrt{x} \sin x, 22) f'(x) = \frac{4}{x}, 23) f'(x) = \cos x - \sin x, 24) f'(x) = x^3 \cos x + 3x^2 \sin x, 25) f'(x) = \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}, 26) f'(x) = \frac{x(x^4+4) - 4x^3 \sin x}{(x^4+1)^2}, 27) f'(x) = \frac{2}{\sin x + \cos x}, 28) f'(x) = 0, 29) f'(x) = \frac{3}{4\sqrt{x}} - 4026x^{2012}, 30) f'(x) = (10x^9 - 30x^{-11})(\sin x + e^x) + (10^3 + 3x^{-10})(\cos x + e^x), 31) f'(x) = -(x+1)^2 \cdot \frac{1}{\sqrt{x+1}(1-x)}, 32) f'(x) = 3 \left( \frac{\sin x}{x} \right)^2 \left( \frac{1}{1 + \cos x} \right), 33) f'(x) = -12 \cos^2 x \sin x 4x, 34) f'(x) = 30x^2 (2x^3 - 1)^4, 35) f'(x) = 5 \left( \frac{1+x^2}{1+x} \right) \frac{x^4+2x-1}{(1+x)^2}, 36) f'(x) = 1 - \frac{1}{x^2}.

Exercise 3.3. a) \( f' \) for \( x < 1 \), and for \( x > 2 \), \( f \) \( \nearrow \) for \( x \in (1, 2) \), b) \( f' \) for \( x < -2 \), and for \( x > \frac{3}{4} \), \( f \searrow \) for \( x \in (-2, \frac{3}{4}) \), c) \( f' \) for \( x < 0 \) and for \( x > 4 \), \( f \searrow \) for \( x \in (0, 2) \) and for \( x \in (2, 4) \), d) \( f' \) for \( x \in \mathbb{R} \), e) \( f' \) for \( x \in (-\frac{7}{3}, 3) \), \( f \searrow \) for \( x < -\frac{7}{3} \), and for \( x > 3 \), f) \( f' \) for \( x > 0 \), \( f \searrow \) for \( x < -\frac{3}{4} \), and for \( x > \frac{1}{2} \), \( f \searrow \) for \( x \in (-\frac{3}{2}, \frac{1}{2}) \), h) \( f' \) for \( x < 0 \), and for \( x > 2 \), \( f \searrow \) for \( x \in (0, 2) \), i) \( f' \) for \( x \in \mathbb{R} \).

Exercise 3.4. a) \( f_{\min}(1) = 2, f_{\max}(-3) = 34, b) f_{\min}(5) = \frac{4}{3}, f_{\max}(1) = \frac{44}{9}, c) f_{\min}(0) = \frac{1}{4}, d) f_{\min}(2) = -3, f_{\max}(4) = 1, e) f_{\min}(-1) = -3, f_{\max}(1) = 1, f) \text{brak}, g) f_{\min}(0) = -11, f_{\max}(1) = -10, f_{\min}(2) = -11, h) \text{brak}, i) f_{\min}(0) = 2.

Exercise 2.4. a) the largest: \( f(-2) = 80 \), the smallest: \( f(2) = -48 \), b) the largest: \( f(1) = -1 \), the smallest: \( f(-1) = -15 \), c) the smallest: \( f(4) = -52 \), the largest: \( f(1) = 2 \), d) the smallest: \( f\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = -\frac{1}{2} \), the largest: \( f(2) = 12 \).

Exercise 3.5. a) convex: \( x \in (1, \infty) \) and \( x \in (-\infty, -1) \), concave: \( x \in (-1, 1) \), the point of inflexion: \( x = 1 \) and \( x = -1 \), b) convex: \( x \in (-1, \infty) \), concave \( x \in (-\infty, -1) \), there are not points of inflexion, c) convex: \( x \in (1, \infty) \) and \( x \in (-\sqrt{3} - 2, \sqrt{3} - 2) \), concave: \( x \in (\sqrt{3} - 2, 1) \) and \( x \in (-\infty, -\sqrt{3} - 2) \), the point of inflexion: \( x = -\sqrt{3} - 2 \) and \( x = \sqrt{3} - 2 \), d) convex: \( x \in (2, \infty) \), concave \( x \in (-\infty, 2) \), the point of inflexion: \( x = 2 \), e) convex: \( x \in (-\frac{3}{4} + k\pi, \frac{3}{4} + k\pi) \), concave: \( x \in \left(\frac{3}{4} + k\pi, \frac{3}{4} + k\pi\right) \), the point of inflexion: \( x = \frac{3}{4} + k\pi, k \in \mathbb{Z} \), f) convex in the domain of function.

Exercise 3.7. a) \( \frac{3}{2} \), b) \( \frac{1}{2} \), c) \( \frac{1}{e} \), d) \( \frac{1}{2} \), e) \( \infty \), f) 1, g) 0, h) 1, i) 0, j) 1, k) 0, l) 1.
Chapter 3

Integration

3.1 Indefinite integral

In this chapter we will examine the reversal of the process of differentiation - integration.

**Definition 3.1.** A function $F$ is said to be an antiderivative of function $f$ if $F' = f$.

**Example 3.1.** Find the antiderivative of $f(x) = 4x^3 + 3x^2 + 2x - 4$.

**Solution.** Let $F(x) = x^4 + x^3 + x^2 - 4x$, so $F'(x) = f(x)$ and $F$ is antiderivative of $f$.

Let us notice, that $F(x) = x^4 + x^3 + x^2 - 4x + 2$, is also antiderivative of $f$.

**Definition 3.2.** The family of all antiderivatives of function $f$ is called indefinite integral of $f$ and is denoted $\int f(x) \, dx$. So

$$\int f(x) \, dx = F(x) + C$$

In the indefinite integral we also denoted

- the function $f$ we call integrand
- $x$ variable of integration
- $C$ constant of integration
- $F$ antiderivatives of function $f$

**Example 3.2.**

$$\int (4x^3 + 3x^2 + 2x - 4) \, dx = x^4 + x^3 + x^2 - 4x + C$$
Now we present the relationship between differentiation and antidifferentiation.

**Theorem 3.1.** If $f$ is differentiable then

$$\int f'(x) \, dx = f(x) + c$$

### 3.1.1 Basic rules for integrating

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int 0 , dx$</td>
<td>$c$</td>
</tr>
<tr>
<td>$\int x^p , dx$</td>
<td>$\frac{1}{p+1} x^{p+1}$</td>
</tr>
<tr>
<td>$\int e^x , dx$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\int \sin x , dx$</td>
<td>$-\cos x + c$</td>
</tr>
<tr>
<td>$\int x , dx$</td>
<td>$x + c$</td>
</tr>
<tr>
<td>$\int \ln x , dx$</td>
<td>$\frac{a^x}{\ln a} + c$</td>
</tr>
<tr>
<td>$\int \cos x , dx$</td>
<td>$\sin x + c$</td>
</tr>
</tbody>
</table>

**Example 3.3.**

$$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{1}{\frac{1}{2} + 1} x^{\frac{1}{2} + 1} + C = \frac{2}{3} \sqrt{x^3} + C$$

Now we present algebraic rules for indefinite integration

**Theorem 3.2.** If there exists antiderivatives of functions $f$ and $g$, then

1. $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$,
2. $\int (f(x) - g(x)) \, dx = \int f(x) \, dx - \int g(x) \, dx$,
3. $\int (cf(x)) \, dx = c \int f(x) \, dx$

**Example 3.4.**

$$\int (3x^2 - 2x + x^4 - 22) \, dx = \int 3x^2 \, dx - \int 2x \, dx + \int x^4 \, dx - \int 22 \, dx =$$

$$= 3 \int x^2 \, dx - 2 \int x \, dx + \int x^4 \, dx - 22 \int dx =$$

$$= \frac{3x^3}{3} - 2 \frac{x^2}{2} + \frac{x^5}{5} - 22x + C =$$

$$= x^3 - x^2 + \frac{x^5}{5} - 22x + C$$
\[
\int \left(7x^6 - 2x^3 + 9x - \sqrt{5} \right) \frac{dx}{x^2} = \int 7x^4 \, dx - \int 2x \, dx + \int \frac{9}{x} \, dx - \int \frac{\sqrt{5}}{x^2} \, dx = \\
= 7 \int x^4 \, dx - 2 \int x \, dx + 9 \int \frac{1}{x} \, dx - \sqrt{5} \int x^{-2} \, dx = \\
= 7 \frac{x^5}{5} - 2 \frac{x^2}{2} + 9 \ln x - \sqrt{5} \frac{x^{-1}}{-1} + C = \\
= 7 \frac{x^5}{5} - x^2 + 9 \ln x + \frac{\sqrt{5}}{x} + C
\]

**Theorem 3.3.** If \( f \) is continuous and \( F \) is antiderivative of \( f \) and \( \varphi \) is continuous then

\[
\int f(x) \, dx = \int f(\varphi(t)) \varphi'(t) \, dt = F(\varphi(t)) + C
\]

**Example 3.5.**

\[
\int \sin^7 x \cos x \, dx \quad t = \sin x \quad dt = \cos x \, dx
\]

Now we present some useful properties of integration

**Theorem 3.4.** If there exists antiderivatives of functions \( f \) and \( g \), then

\[
\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C \\
\int \frac{f'(x)}{\sqrt{f(x)}} \, dx = 2 \sqrt{f(x)} \, dx + C
\]

**Example 3.6.**

\[
\int \frac{2x^2 - 4x + 3}{\frac{2}{3}x^3 - 2x^2 + 3x - 17} \, dx = \ln \left| \frac{2}{3}x^3 - 2x^2 + 3x - 17 \right| + C \\
\int \frac{\sin x + 2}{\sqrt{-\cos x + 2x + 12}} \, dx = 2 \sqrt{-\cos x + 2x + 12} + C
\]

### 3.2 Definite integral

**Definition 3.3.** If the function \( f \) is continuous and

\[
\int f(x) \, dx = F(x) + C
\]
then
\[ \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \]

where

- \(a\) is called lower limits of integration
- \(b\) is called upper limits of integration

**Example 3.7.**

\[ 3 \int_1^3 \sqrt{x}\,dx = \int_1^3 x^{1/2}\,dx = \left[ \frac{2}{3} \sqrt{x^3} \right]_1^3 = \frac{2}{3} \sqrt{3^3} - \frac{2}{3} \sqrt{1^3} = 2\sqrt{3} - \frac{2}{3} \]

There are some basic properties of the definite integral. Some of these properties are similar to those for the indefinite integral.

**Theorem 3.5.** If \(f\) and \(g\) are continuous, then

1. \( \int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \)
2. \( \int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \)
3. \( \int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx, \) gdzie \(c \in \mathbb{R}\)
4. \( \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \)
5. \( \int_a^b f(x)\,dx = 0 \)
6. \( \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \)

**Example 3.8.**

\[ \int_0^1 (x^3 + 3x + 1)\,dx = \left[ \frac{1}{4}x^4 + \frac{3}{2}x^2 + x \right]_0^1 = \frac{1}{4}1^4 + \frac{3}{2}1^2 + 1 - (\frac{1}{4}0^4 + \frac{3}{2}0^2 + 0) \]

\[ = \frac{11}{4} - 0 = 2\frac{2}{3} \]

The subdivision rule for definite integrals
**Theorem 3.6.** If $f$ is integrable on $[a, b]$ and $c \in (a, b)$, then

$$
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
$$

**Example 3.9.** Evaluate $\int_0^3 |x^2 - 2x| \, dx$

**Solution.** Because of definition of absolute value, we have

$$
|x^2 - 2x| = \begin{cases} 
-x^2 + 2x & \text{dla } 0 \leq x \leq 2 \\
x^2 - 2x & \text{dla } 2 < x \leq 3 
\end{cases}
$$

Let us divide the integral $[0, 3]$ on $[0, 2]$ and $[2, 3]$. Then

$$
\int_0^3 |x^2 - 2x| \, dx = \int_0^2 |x^2 - 2x| \, dx + \int_2^3 |x^2 - 2x| \, dx
$$

$$
= \int_0^2 (-x^2 + 2x) \, dx + \int_2^3 (x^2 - 2x) \, dx
$$

$$
= \left[-\frac{1}{3}x^3 + x^2\right]_0^2 + \left[\frac{1}{3}x^3 - x^2\right]_2^3
$$

$$
= -\frac{8}{3} + 4 - 0 + 9 - 9 - \left(\frac{8}{3} - 4\right) = -\frac{16}{3} + 8 = \frac{8}{3}
$$

**3.2.1 The geometric interpretation of the definite integral**

If $f$ is non-negative on the interval $[a, b]$ then the definite integral represents the area of the region bounded by the curve and the $x$-axis over this interval.
Chapter 3. Integration

\[ P = \int_{a}^{b} f(x)dx \]

If the function is negative over \([a, b]\), then \(\int_{a}^{b} f(x)dx = -P\)

Example 3.10. Find the area of the region enclosed by the curves \(\frac{1}{2}x^2\) and \(y = 2x\).

Solution.
First, we must to find the points where the curves intersect.

\[
\begin{cases}
  y = \frac{1}{2}x^2 \\
  y = 2x
\end{cases}
\]

so

\[
\begin{align*}
  \frac{1}{2}x^2 &= 2x \\
  \frac{1}{2}x^2 - 2x &= 0 \\
  x \left( \frac{1}{2}x - 2 \right) &= 0 \\
  x &= 0 \quad \text{or} \quad x = 4
\end{align*}
\]

therefore the corresponding points \((0, 0)\) and \((4, 8)\) are the only point of intersection. The region enclosed by two curves is bounded above by \(y = 2x\) and below by \(y = \frac{1}{2}x^2\) over the interval
[0, 4]. So the area of this region is given by the integral

\[
P = \int_{0}^{4} \left[ 2x - \frac{1}{2}x^2 \right] dx = \int_{0}^{4} 2xdx - \int_{0}^{4} \frac{1}{2}x^2dx =
\]

\[
= 2 \int_{0}^{4} xdx - \frac{1}{2} \int_{0}^{4} x^2dx = 2 \left[ \frac{1}{2}x^2 \right]_{0}^{4} - \frac{1}{2} \left[ \frac{1}{3}x^3 \right]_{0}^{4} =
\]

\[
= 2 \left[ \frac{1}{2}4^2 - \frac{1}{2}0^2 \right] - \frac{1}{2} \left[ \frac{1}{3}4^3 - \frac{1}{3}0^3 \right] =
\]

\[
= 2 \cdot \frac{1}{2}4^2 - \frac{1}{2} \cdot \frac{1}{3}4^3 = \frac{16}{3}
\]
3.3 Review Exercises

Exercise 3.1. Evaluate the given integral.

a) \( \int \frac{dx}{x^5} \)  

b) \( \int \frac{dx}{x^{101}} \)

c) \( \int (x + 3)(x^2 + 1) \, dx \)  

d) \( \int \frac{2x^2 + x - 1}{x} \, dx \)

e) \( \int (2x^{-1.3} + 4x^{-0.7} + 6x^{0.21}) \, dx \)  

f) \( \int e^x \left(1 - \frac{e^x}{x^2}\right) \, dx \)

g) \( \int (7x^6 - 4x^3 + 3 + \sin x) \, dx \)  

h) \( \int (5 \sin x - 6 \cos x) \, dx \)

i) \( \int \frac{1}{\cos^2 x} + \sin x \, dx \)  

j) \( \int \frac{\sqrt{x^4 + x^3}}{x^4} \, dx \)

k) \( \int (x^5 + \frac{4}{5}x^3 + 6) \, dx \)  

l) \( \int \frac{x^6 + 3x^4 + 6}{x^5} \, dx \)

m) \( \int \tan x \, dx \)

n) \( \int \cot x \, dx \)  

o) \( \int \frac{\ln^2}{\ln 2x - \ln x} \, dx \)

Exercise 3.2. Evaluate the given integral.

a) \( \int \frac{x}{(x^2 + 6)} \, dx \)  

b) \( \int \frac{3x^2}{x^3 + 4} \, dx \)

c) \( \int \sqrt{1 + x^2} \, dx \)  

d) \( \int \frac{x}{\sqrt{3 - 5x^2}} \, dx \)

e) \( \int e^{-4x + 7} \, dx \)  

f) \( \int e^{x^4} x^3 \, dx \)

g) \( \int \frac{2x + 2}{\sqrt{x^2 + 2x - 1}} \, dx \)  

h) \( \int (3x^4 + 12x)^3 (x^3 + 1) \, dx \)

i) \( \int x \sin (2x^2 + 8) \, dx \)  

j) \( \int \sin \frac{x}{2} \, dx \)

k) \( \int \frac{dx}{x(3 + \ln x)} \)  

l) \( \int (2x + 7)^5 \, dx \)

Exercise 3.3. Evaluate the given integral.
Chapter 3. Integration

Exercise 3.4. Evaluate the given integral.

a) $\int_{0}^{6} 5x \, dx$

b) $\int_{0}^{10} 3x^2 \, dx$

c) $\int_{-1}^{3} (x + 3) \, (x + 1) \, dx$

d) $\int_{-\frac{4}{3}}^{1} x^{-\frac{3}{2}} \, dx$

e) $\int_{\frac{1}{3}}^{3} (x^3 + x + 6) \, dx$

f) $\int_{1}^{10} \left( \frac{1}{\sqrt{x} + \sqrt{x}} \right) \, dx$

g) $\int_{0}^{1} 4e^{2x} \, dx$

h) $\int_{0}^{\frac{\pi}{2}} 2e^{-2x} \, dx$

i) $\int_{0}^{\pi} \sin x \, dx$

j) $\int_{0}^{\frac{\pi}{2}} \cos x \, dx$

k) $\int_{0}^{1} (\sin x + \cos x) \, dx$

l) $\int_{1}^{2} \left( \frac{x^3}{7} + \frac{2}{3x} \right) \, dx$

m) $\int_{0}^{2} (2x^2 - \sqrt{x} + 1) \, dx$

n) $\int_{0}^{\frac{2}{3}} \left( \frac{2}{3} x^{-\frac{5}{2}} + 4\sqrt{x} - 7 \right) \, dx$

o) $\int_{-1}^{8.5} (3x^3 - 2.5x) \, dx$

Exercise 3.5. Find the area of the region enclosed by the curves $y = x^2 - 4x + 4$, $x = 0$, $x = 1$ and x-axis.

Exercise 3.6. Find the area of the region enclosed by the curves $y = x^2$ and $y = 2x$.

Exercise 3.7. Find the area of the region enclosed by the curves $y = e^x$, $y = -x + 1$ and $x = 1$.

Exercise 3.8. Find the area of the region enclosed by the curves $y = x^3$, $y = 4x$ and y-axis.
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Answers

Exercise 3.1
a) \( -\frac{1}{4}x^{-4} + C \), b) \(-\frac{1}{100}x^{-100}, c) 3x + \frac{1}{2}x^2 + x^3 + \frac{1}{4}x^4 + C \), d) \( x^2 + x - \ln x + C, e) -\frac{20}{3}x^{-0.3} + \frac{40}{3}x^{0.3} + 5x^5 + C \), f) \( e^x + x^{-1} + C \), g) \( x^7 - x^4 + 3x - \cos x + C \), h) \(-5 \cos x - 6 \sin x + C \), i) \( \tan x - \cos x + C \), j) \( e^x + x - 1 + C \), k) \( \frac{1}{6}x^6 + \frac{1}{3}x^4 + 6x + C \), l) \( \frac{1}{4}x^4 - 3x^{-1} - \frac{3}{2}x^{-4} + C \), m) \( \sin x + 3 \frac{x^4}{x^3} + C \), n) \( \ln x + C, o) x \)

Exercise 3.2
a) \( \frac{1}{2} \ln (x^2 + 6) \), b) \( \ln (x^3 + 4) + C \), c) \( \frac{1}{3} \sqrt{3 - 5x^2} + C \), d) \(-\frac{1}{3} \sqrt{3 - 5x^2} + C \), e) \(-\frac{1}{4}e^{-4x^2} + C \), f) \( 2\sqrt{2x + x^2 - 7} + C \), g) \( \frac{1}{18} \left( 3x^4 + 12x \right) + C \), h) \(-\frac{1}{4} \cos (2x^2 + 8) + C \), i) \(-\frac{5}{2} \cos \left( \frac{7}{5}x + C \right) \)

Exercise 3.3
a) \( 90, b) 999, c) 18, d) 9, e) 36, f) 2e^6 - 2, h) 1 - e^{-20}, i) 2, j) 1, k) 2, l) \( \frac{2}{3} \ln 2 + \frac{15}{28}, l) \frac{2}{3} \ln 3 \), m) \( \frac{22}{3} - \frac{4}{3} \sqrt{2}, n) 4 \sqrt{2} - \frac{20}{3} \), o) 28.125.

Exercise 3.4
a) \( 16, b) \frac{14}{9} \sqrt{7}, c) \frac{28}{9}, d) 3 \ln 10, e) 2e^6 - 2e^3, f) \frac{1}{2}e^3 - \frac{1}{2}e \)

Exercise 3.5 \( 2 \frac{2}{3} \)

Exercise 3.6 \( \frac{1}{3} \)

Exercise 3.7 \( e - \frac{3}{2} \)

Exercise 3.8 \( 4 \)
Chapter 4

Introduction to algebra

4.1 Sigma notation

Consider sequences \( a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots \) and \( c_1, c_2, c_3, \ldots \). For example

- the sum \( a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \) we can write in convenient form \( \sum_{i=1}^{6} a_i \),

- the sum \( b_1 + b_2 + \ldots + b_{12} \) we can write as \( \sum_{k=1}^{12} b_k \),

- the sum \( c_1 + c_2 + \ldots + c_m \) we can write as \( \sum_{l=1}^{m} c_l \).

**Definition 4.1.** The sum \( a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_n \) is written in sigma notation as \( \sum_{i=1}^{n} a_i \).

**Remark 4.1.** The symbol \( \sum \) (read: sigma) is generally used to denote a sum of multiple terms. This symbol is generally accompanied by an index. Always, the index must be considered in the sum.

**Example 4.1.** Evaluate

- \( \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 \)

- \( \sum_{k=0}^{4} 2k = 2 \cdot 0 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 = 0 + 2 + 4 + 6 + 8 = 20 \)

- \( \sum_{k=1}^{6} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{49}{20} \)
• \( \sum_{i=1}^{6} 2^i = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2 \frac{1-6^6}{1-2} = 2 \cdot 63 = 126 \)

• \( \sum_{i=1}^{n} aq^i = aq^1 + aq^2 + ... + aq^n = aq \frac{1-q^n}{1-q} \)

• \( \sum_{i=5}^{7} (3i + 1) = (3 \cdot 5 + 1) + (3 \cdot 6 + 1) + (3 \cdot 7 + 1) = 16 + 19 + 22 = 57 \)

• \( \sum_{k=1}^{8} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{1+8}{2} \cdot 8 = 36 \)

• \( \sum_{k=1}^{20} k = 10 + 11 + ... + 20 = \frac{10 + 20}{2} \cdot 11 = 165 \)

• \( \sum_{k=1}^{m} k = 1 + 2 + ... + m = \frac{1+m}{2} \cdot m \)

• \( \sum_{k=l}^{m} k = l + ... + m = \frac{l+m}{2} \cdot (m - l + 1) \)

Rules for sigma notation

**Theorem 4.1.** Basic rules for use with sigma notation are the following

- \( \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)
- \( \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \)
- \( \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \)
- \( \sum_{i=1}^{n} c = c + c + ... + c = nc \)
- \( \sum_{i=m}^{n} c = (n - m + 1)c \)

**Example 4.2.**

\[
\sum_{i=5}^{7} (3i + 1) = \sum_{i=5}^{7} 3i + \sum_{i=5}^{7} 1 = 3 \sum_{i=5}^{7} i + (7 - 5 + 1) \cdot 1 = 3(5 + 6 + 7) + 3 \cdot 1 = 3 \cdot 18 + 3 = 54 + 3 = 57
\]

\[
\sum_{k=2}^{12} 5k = 5 \sum_{k=2}^{12} k = 5 \frac{5 + 12}{2} (12 - 5 + 1) = 5 \frac{17}{2} \cdot 8 = 340
\]
4.1.1 The double sum

**Definition 4.2.** The double sum we call the sum of sum

\[
\sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik}
\]

where we use the index \(i\) from \(i = 1\) to \(n\) and for \(k\) from \(k = 1\) to \(m\).

**Example 4.3.**

\[
\begin{align*}
\sum_{i=1}^{3} \sum_{k=1}^{2} (i + k) &= \sum_{i=1}^{3} [(i + 1) + (i + 2)] \\
&= [(1 + 1) + (1 + 2)] + [(2 + 1) + (2 + 2)] + [(3 + 1) + (3 + 2)] \\
&= 5 + 7 + 9 = 21
\end{align*}
\]

or quicker

\[
\begin{align*}
\sum_{i=1}^{3} \sum_{k=1}^{2} (i + k) &= \sum_{i=1}^{3} [(i + 1) + (i + 2)] = \sum_{i=1}^{3} [2i + 3] = 5 + 7 + 9 = 21
\end{align*}
\]

To represent the data in table or matrices, we often use a double index notation. First index corresponds to the number of the row where the data is located and the second to the column.

**Example 4.4.** Let

\[
\begin{align*}
a_{11} &= 0 & a_{12} &= 4 & a_{13} &= 5 & a_{14} &= 3 \\
a_{21} &= 3 & a_{22} &= -1 & a_{23} &= 1 & a_{24} &= 7 \\
a_{31} &= 2 & a_{32} &= 0 & a_{33} &= 1 & a_{34} &= 0
\end{align*}
\]

Evaluate

\[
\begin{align*}
\sum_{j=1}^{4} a_{2j} &= a_{21} + a_{22} + a_{23} + a_{24} = 3 + (-1) + 1 + 7 = 10 \\
\sum_{i=1}^{3} a_{i1} &= a_{11} + a_{21} + a_{31} = 0 + 3 + 2 = 5 \\
\sum_{i=1}^{3} \sum_{j=2}^{3} a_{ij} &= a_{12} + a_{13} + a_{22} + a_{23} + a_{32} + a_{33} = 4 + 5 + (-1) + 1 + 0 + 1 = 10
\end{align*}
\]
Rules for double sigma notation

**Theorem 4.2.** Basic rules for use with double sigma notation are the following

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} (a_{ik} + b_{ik}) = \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} + \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik} \)

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} (a_{ik} - b_{ik}) = \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} - \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik} \)

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} ca_{ik} = c \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} \)

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} = \sum_{k=1}^{m} \sum_{i=1}^{n} a_{ik} \)

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} a_{i} = m \sum_{i=1}^{n} a_{i} \)

- \( \sum_{i=1}^{n} \sum_{k=1}^{m} a_{k} = n \sum_{k=1}^{m} a_{k} \)

**Example 4.5.** For statistical purposes the school in the region are numbered from 1 to 100. By \( x_{ik} \) we have been marked graduates who have completed the \( i \)-th universities in the \( k \)-th month. What do the following sums mean?

1. \( \sum_{k=1}^{6} x_{17k} \)
2. \( \sum_{i=10}^{20} x_{i6} \)
3. \( \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \)
4. \( \frac{1}{100} \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \)
5. \( \frac{1}{12} \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \)

**Solution.**

1. \( \sum_{k=1}^{6} x_{17k} \) graduates who have completed a university No. 17 during the first half a year

2. \( \sum_{i=10}^{20} x_{i6} \) graduates who have completed college in June, with numbers from 10 to 20
3. \( \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \) all university graduates who have completed it within the year

4. \( \frac{1}{100} \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \) average number of graduates per university during the whole year

5. \( \frac{1}{12} \sum_{i=1}^{100} \sum_{k=1}^{12} x_{ik} \) average monthly number of graduates

### 4.2 Matrix

**Definition 4.3.** The matrix of order \( m \times n \), where \( m, n \in \mathbb{N} \) we defined as an array of numbers (or algebraic symbols) set out in rows and columns.

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\]

Matrices will be denoted by capital letters: \( A, B, C, X \).

Matrix element standing in the \( i \)-th row and \( j \)-th column is denoted by \( a_{ij} \).

So we can write \( A = [a_{ij}]_{m \times n} \).

**Example 4.6.**

- \( A = \begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix} \) – matrix of order \( 2 \times 2 \)

- \( B = \begin{bmatrix} 2 & 3 & 1 & \frac{1}{2} \end{bmatrix} \) – matrix of order \( 1 \times 4 \)

- \( C = \begin{bmatrix} \\
2 & 3 & 7 \\
1 & 2 & 5 \\
0 & 3 & 3 \\
0 & 4 & 2 \\
\end{bmatrix} \) – matrix of order \( 4 \times 3 \) and: \( c_{12} = 3, c_{23} = 5 \)

- \( D = \begin{bmatrix} -2 \\ 10 \\ 0.7 \end{bmatrix} \) – matrix of order \( 3 \times 1 \)
**Definition 4.4.** A square matrix of order \( n \) is a matrix with the same number of rows and columns \( m = n \).

Elements \((a_{11}, a_{22}, \ldots, a_{nn})\) we called the main diagonal.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{12} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

**Example 4.7.**

\[
A = \begin{bmatrix}
2 & 2 & 7 \\
1 & -4 & 5 \\
0 & 2 & 0
\end{bmatrix}
\]

is the matrix of order 3. The main diagonal is of the form \( a_{11} = 2, a_{22} = -4, a_{33} = 0 \).

### 4.2.1 Types of matrices

**Definition 4.5.** The square matrix of order \( n \), with all entries outside the main diagonal are equal to zero, we call diagonal matrix.

\[
D = \begin{bmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\]

**Example 4.8.** The diagonal matrix of order 5

\[
D_5 = \begin{bmatrix}
9 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**Definition 4.6.** The diagonal matrix, with all entries on the main diagonal are equal one is
called the identity matrix and denoted by $I_n$.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

**Example 4.9.** The identity matrix of order 2 is

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the identity matrix of order 5

$$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Definition 4.7.** The square matrix $n \geq 2$, for which all the entries above the main diagonal are 0 is said to be lower triangular.

$$
T = \begin{bmatrix}
    a_{11} & 0 & \cdots & 0 \\
    a_{12} & a_{22} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
$$

The square matrix $n \geq 2$, for which all the entries below the main diagonal are 0 is said to be lower triangular.

$$
T = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    0 & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{nn}
\end{bmatrix}
$$

**Example 4.10.** A lower triangular matrix:

$$
T_3 = \begin{bmatrix}
    12 & 0 & 0 \\
    -45 & 21 & 0 \\
    2 & 2 & 2
\end{bmatrix}
$$
An upper triangular matrix:

$$T_4 = \begin{bmatrix}
2 & 4 & 3 & 1 \\
0 & 1 & 8 & -4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

### 4.2.2 Elementary Matrix Operations

#### Definition 4.8. Two matrices $A$ and $B$ are equal when they have the same orders and the following condition holds

$$a_{ij} = b_{ij} \text{ for } i \in \{1, 2, \ldots, m\}, \ j \in \{1, 2, \ldots, n\}$$

#### Example 4.11. The matrices

$$A = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} \text{ and } B = \begin{bmatrix}
1 \\
0
\end{bmatrix}$$

are equal $A = B$.

#### Example 4.12. The matrices

$$A = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} \text{ and } C = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix}$$

are not equal $A \neq C$, because they have different orders.

#### Example 4.13. The matrices

$$A = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} \text{ and } D = \begin{bmatrix}
1 \\
0
\end{bmatrix}$$

are not equal $A \neq D$, because $a_{21} \neq d_{21}$

#### Definition 4.9. Let $A = [a_{ij}]_{m \times n}$. The transpose of a matrix $A$ is another matrix denoted by $B = [b_{ij}]_{n \times m}$ and

$$b_{ij} = a_{ji},$$

for $i \in \{1, 2, \ldots, m\}, \ j \in \{1, 2, \ldots, n\}$. The transpose matrix we denoted by $B = A^T$. 
Remark 4.2. The rows of $A^T$ are the columns of $A$, and the rows of $A$ are the columns of $A^T$.

Example 4.14. Let

$$A = \begin{bmatrix} -5 & 2 & 1 \\ -1 & 6 & 2 \\ 2 & 0 & 1 \\ 2 & 64 & 5 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} -5 & -1 & 2 & 2 \\ 2 & 6 & 0 & 64 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

Obviously the transpose of $A^T$ is again $A$, so

$$(A^T)^T = A$$

Definition 4.10. Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$.

The sum (difference) of two matrices of the same order, $A$ and $B$, is written $A \pm B$ and defined to be the matrix $C = [c_{ij}]_{m \times n}$, for which

$$c_{ij} = a_{ij} \pm b_{ij},$$

for $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$.

In this case we have

$$C = A \pm B.$$
find $A + A$ and $A - A$.

Solution.

$$A + B = \begin{bmatrix}
3 + \frac{2}{3} & 1 + 1 & 0 - 3 & 1 + 4 \\
\frac{2}{3} + 14 & -5 + 9 & 1 + 11 & 4 + 0 \\
2 - 4 & -4 + 3 & 2 \frac{1}{2} + 1 & 3 - 2
\end{bmatrix} = \begin{bmatrix}
\frac{11}{3} & 2 & -3 & 5 \\
44 & 4 & 12 & 4 \\
-2 & -1 & \frac{7}{2} & 1
\end{bmatrix}$$

and

$$A - B = \begin{bmatrix}
3 - \frac{2}{3} & 1 - 1 & 0 + 3 & 1 - 4 \\
\frac{2}{3} - 14 & -5 - 9 & 1 - 11 & 4 - 0 \\
2 + 4 & -4 - 3 & 2 \frac{1}{2} - 1 & 3 + 2
\end{bmatrix} = \begin{bmatrix}
\frac{7}{3} & 0 & 3 & -3 \\
-\frac{40}{3} & -14 & -10 & 4 \\
6 & -7 & \frac{3}{2} & 5
\end{bmatrix}$$

Definition 4.11. Let $A = [a_{ij}]_{m \times n}$, $\alpha$ be a scalar. Multiplication of a matrix $A$ by a scalar $\alpha$ is another matrix $B = [b_{ij}]_{m \times n}$ which is defined by

$$b_{ij} = \alpha \cdot a_{ij},$$

for $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$.

Then we can write

$$B = \alpha \cdot A.$$

and

$$\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mn}
\end{bmatrix} = \begin{bmatrix}
\alpha \cdot a_{11} & \alpha \cdot a_{12} & \cdots & \alpha \cdot a_{1n} \\
\alpha \cdot a_{21} & \alpha \cdot a_{22} & \cdots & \alpha \cdot a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha \cdot a_{m1} & \alpha \cdot a_{m2} & \cdots & \alpha \cdot a_{mn}
\end{bmatrix}$$

Example 4.16. Consider the matrix

$$A = \begin{bmatrix}
1 & 4 \\
-2 & 1
\end{bmatrix}.$$

find $3 \cdot A$.

Solution.

$$3 \cdot A = 3 \cdot \begin{bmatrix}
1 & 4 \\
-2 & 1
\end{bmatrix} = \begin{bmatrix}
3 \cdot 1 & 3 \cdot 4 \\
3 \cdot (-2) & 3 \cdot 1
\end{bmatrix} = \begin{bmatrix}
3 & 12 \\
-6 & 3
\end{bmatrix}$$
Example 4.17. Consider matrices

\[ A = \begin{bmatrix} 5 & 6 & 2 \\ -5 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -7 & 1 \\ 4 & 1 & 3 \end{bmatrix} \]

find

\[ 3 \cdot A - 4 \cdot B \]

Solution.

\[ 3 \cdot A - 4 \cdot B = 3 \cdot \begin{bmatrix} 5 & 6 & 2 \\ -5 & 2 & -1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 3 & -7 & 1 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 18 & 6 \\ -15 & 6 & -3 \end{bmatrix} - \begin{bmatrix} 12 & -28 & 4 \\ 16 & 4 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 46 & 2 \\ -31 & 2 & -15 \end{bmatrix} \]

Example 4.18. Find the matrix \( X \).

\[ X - 4 \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = 2I - X \]

Solution. Of course the matrix \( I \) is a matrix of order 3. So

\[ X - 4 \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - X \]

and

\[ X + X = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 4 \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \]

\[ 2X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 12 & 16 \\ 4 & 8 & 12 \\ 0 & 4 & 8 \end{bmatrix} \]

\[ 2X = \begin{bmatrix} 6 & 12 & 16 \\ 4 & 10 & 12 \\ 0 & 4 & 10 \end{bmatrix} \]
and finally
\[
X = \frac{1}{2} \begin{bmatrix}
6 & 12 & 16 \\
4 & 10 & 12 \\
0 & 4 & 10
\end{bmatrix}
\]
\[
X = \begin{bmatrix}
3 & 6 & 8 \\
2 & 5 & 6 \\
0 & 2 & 5
\end{bmatrix}
\]

Definition 4.12. Let \( A = [a_{ij}]_{m \times n} \), \( B = [b_{ij}]_{n \times k} \).

Product of two matrices \( A \) and \( B \) is a matrix \( C = [c_{ij}]_{m \times k} \), denoted by the formula
\[
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}
\]
for \( i \in \{1, 2, \ldots, m\} \), \( j \in \{1, 2, \ldots, k\} \).

In this case we can write
\[
C = A \cdot B.
\]

- Product \( A \cdot B \) is feasible if the number of columns matrix \( A \) is the same as the number of rows of the matrix \( B \).
- Product of two matrices \( A \) and \( B \) has as many rows as the matrix \( A \) and as many columns as the matrix \( B \).
- Usually matrix multiplication is not commutative.

Example 4.19. Let
\[
A = \begin{bmatrix}
1 & 3 & 12 \\
-2 & 21 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
5 & 2 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\]
find \( A \cdot B \) and \( B \cdot A \).

Solution.
\[
AB = \begin{bmatrix}
1 \cdot 5 + 3 \cdot 1 + 12 \cdot 1 & 1 \cdot 2 + 3 \cdot 0 + 12 \cdot 1 \\
-2 \cdot 5 + 21 \cdot 1 + (-2) \cdot 1 & -2 \cdot 2 + 21 \cdot 0 + (-2) \cdot 1
\end{bmatrix} = \begin{bmatrix}
20 & 14 \\
9 & -6
\end{bmatrix}
\]
and

\[
BA = \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 12 \\ -2 & 21 & -2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 2 \cdot (-2) & 5 \cdot 3 + 2 \cdot 21 & 5 \cdot 12 + 2 \cdot (-2) \\ 1 \cdot 1 + 0 \cdot (-2) & 1 \cdot 3 + 0 \cdot 21 & 1 \cdot 12 + 0 \cdot (-2) \\ 1 \cdot 1 + 1 \cdot (-2) & 1 \cdot 3 + 1 \cdot 21 & 1 \cdot 12 + 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 1 & 57 & 56 \\ 1 & 3 & 12 \\ -1 & 24 & 10 \end{bmatrix}
\]

**Definition 4.13.** The product \( AA \) is called the square of \( A \) and is denoted by \( A^2 \). Similarly, \( n \)-th power of the matrix \( A \) is of the form

\[
A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_n
\]

**Example 4.20.** Let

\[
A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}
\]

then

\[
A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix}
\]

\[
A^3 = \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 21 \\ 0 & 8 \end{bmatrix}
\]

\[
A^4 = \begin{bmatrix} 1 & 21 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 45 \\ 0 & 16 \end{bmatrix}
\]

**Example 4.21.** Two factories in the company "ABCboats" produce three types of canoes: single, double and triple. The indicators of working time and cost of production are given by the following matrices: working time (hours)

\[
M = \begin{bmatrix}
\text{cutting} & \text{connecting} & \text{painting} \\
\text{single} & 0.6 & 0.6 & 0.2 \\
\text{double} & 1.0 & 0.9 & 0.3 \\
\text{triple} & 1.5 & 1.2 & 0.4
\end{bmatrix}
\]
Chapter 4. Introduction to algebra

The cost of production (thousand zl)

\[
N = \begin{bmatrix}
\text{cuting} & \text{branch I} & \text{branch II} \\
0.6 & 6 \\
0.2 & 7 \\
\end{bmatrix}
\]

1. The cost of production single canoe produced in the factory I:

\[
\begin{bmatrix}
0.6 \\
0.6 \\
\end{bmatrix}
\begin{bmatrix}
6 \\
8 \\
3 \\
\end{bmatrix} = [9]
\]

2. The cost of production triple canoe produced in the factory II:

\[
\begin{bmatrix}
1.5 \\
1.2 \\
0.4 \\
\end{bmatrix}
\begin{bmatrix}
7 \\
10 \\
4 \\
\end{bmatrix} = [24.1]
\]

3. product \( MN \) is a matrix of the cost of production of different types of boats in factories I and II.

\[
\begin{bmatrix}
0.6 & 0.6 & 0.2 \\
1.0 & 0.9 & 0.3 \\
1.5 & 1.2 & 0.4 \\
\end{bmatrix}
\begin{bmatrix}
6 \\
8 \\
3 \\
\end{bmatrix} = \begin{bmatrix}
\text{single} & 9.0 & 11.0 \\
\text{double} & 14.1 & 17.2 \\
\text{triple} & 19.8 & 24.1 \\
\end{bmatrix}
\]

Now we present and check some basic rules of matrices operations.

**Theorem 4.3.** 1. Let us consider \( A \) of order \( m \times n \) and matrices \( B \) and \( C \) of orders \( n \times k \), then

\[
A (B + C) = AB + AC.
\]

2. Let us consider \( A \) and \( B \) of orders \( m \times n \) and \( C \) of order \( n \times k \), then

\[
(A + B) C = AC + BC.
\]

3. Let us consider the matrix \( A \) of order \( m \times n \), the matrix \( B \) of order \( n \times k \) and any scalar \( \alpha \), then

\[
A (\alpha B) = (\alpha A) B = \alpha (AB)
\]
4. Let us consider the matrix $A$ of order $m \times n$, the matrix $B$ of order $n \times k$ and the matrix $C$ of order $k \times l$, then

$$(AB)C = A(BC).$$

**Example 4.22.** Let us consider

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

verify the second fact in the above theorem.

**Solution.** Our properties is of the form

$$(A + B)C = AC + BC.$$

So

$$A + B = \begin{bmatrix} 3 & 1 \\ 9 & 0 \end{bmatrix}$$

and

$$(A + B)C = \begin{bmatrix} 3 & 1 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 0 \end{bmatrix}$$

On the other hand we have

$$AC = \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 1 & -4 \\ 5 & -2 \end{bmatrix}$$

therefore

$$AC + BC = \begin{bmatrix} 3 & 2 \\ 9 & 0 \end{bmatrix}$$

So our fact holds.

Basic rules of the matrix transposition.

**Theorem 4.4.** 1. Let us consider matrices $A$ and $B$ of order $m \times n$, then

$$(A + B)^T = A^T + B^T.$$

2. Let us consider the matrix $A$ of order $m \times n$ and $\alpha \in \mathbb{R}$, then

$$(A^T)^T = A \quad \text{and} \quad (\alpha A)^T = \alpha A^T$$
3. Let us consider the matrix $A$ of order $m \times n$, the matrix $B$ of order $n \times k$, then

$$(AB)^T = B^T A^T$$

**Example 4.23.** Let us consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

verify the last fact in the above theorem.

**Solution.** Our rule is of the form

$$(AB)^T = B^T A^T.$$  

So on the left side we have

$$AB = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 10 & 10 \\ 7 & 16 & 16 \end{bmatrix}$$

and

$$(AB)^T = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 10 & 16 \\ 4 & 10 & 16 \end{bmatrix}.$$  

On the right side

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix},$$

and

$$B^T A^T = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 10 & 16 \\ 4 & 10 & 16 \end{bmatrix}.$$  

Therefore our fact holds.
4.3 Introduction to Determinants

For the square matrix we can assign a number that can be calculated from the matrix. So for the square matrix $A$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

we denote this special number by $|A|$ or $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Order of determinant is equal the order of the matrix.

4.3.1 Rules of calculation determinants

The determinant of order 1. If $A$ is a matrix of order $n = 1$ ($A = [a_{11}]$), then

$$\det A = a_{11}$$

the determinant of the matrix $A = [25]$, is equal

$$\det A = 25$$

The determinant of order 2. If $A$ is a matrix of order $n = 2$, then

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

the determinant of the matrix

$$A = \begin{bmatrix} 1 & -8 \\ 1 & 2 \end{bmatrix}$$

is equal

$$\det A = 1 \cdot 2 - 1 \cdot (-8) = 2 + 8 = 10$$
The determinant of order 3. To compute the determinant of a $3 \times 3$ matrix we use the Sarrus’ scheme. The first Sarrus’ method: Write out the first 2 rows of the matrix to the bottom of the third row, so that you have 5 rows and 3 columns. Using the scheme below you can compute determinant.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} - (a_{31} \cdot a_{22} \cdot a_{13} + a_{11} \cdot a_{32} \cdot a_{23} + a_{21} \cdot a_{12} \cdot a_{33})$$

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 3 + 3 \cdot 4 \cdot 7 - (4 \cdot 2 \cdot 9 + 1 \cdot 8 \cdot 6 + 7 \cdot 5 \cdot 3) = 45 + 96 + 84 - (72 + 48 + 105) = 225 - 225 = 0$$

The second Sarrus’ method: Write out the first 2 columns of the matrix to the right of the third column, so that you have 5 columns and 3 rows. Using the scheme below you can compute determinant.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \cdot \begin{vmatrix} a & b \\ g & h \end{vmatrix} = aei + bfg + cdh - (gec + hfa + idb)$$

For the previous matrix we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 3 + 3 \cdot 4 \cdot 7 - (4 \cdot 2 \cdot 9 + 1 \cdot 8 \cdot 6 + 7 \cdot 5 \cdot 3) = 45 + 96 + 84 - (72 + 48 + 105) = 225 - 225 = 0$$

The determinant of order 4 and higher. The determinant of order greater than three can be computed with the the Laplace expansion.
Laplace expansion

Consider the square matrix $A = [a_{ij}]$, $n \geq 2$.

**Definition 4.14.** The $a_{ij}$ cofactor of $A$ is the number defined by

$$D_{ij} = (-1)^{i+j} \det M_{ij},$$

where $M_{ij}$ is minor matrix of $A$, that results from $A$ when row $i$ and column $j$ have been deleted.

**Example 4.24.** Consider the matrix

$$A = \begin{bmatrix}
1 & 3 & -4 & 0 \\
0 & 1 & 1 & 7 \\
3 & 2 & 4 & 3 \\
0 & 8 & 6 & 4
\end{bmatrix}$$

the $a_{34}$ cofactor of $A$ is a number

$$D_{34} = (-1)^{3+4} \det M_{34} = (-1)^7 \begin{vmatrix}
1 & 3 & -4 \\
0 & 1 & 1 \\
0 & 8 & 6
\end{vmatrix} = (-1) \cdot [6 + 0 + 0 - (0 + 8 + 0)] = -2$$

**Example 4.25.** Consider the matrix

$$A = \begin{bmatrix}
-4 & 2 & 1 \\
4 & 2 & 3 \\
2 & 1 & 0
\end{bmatrix}$$

Several cofactors of the matrix $A$.

$$D_{13} = (-1)^{1+3} \cdot \begin{vmatrix}
4 & 2 \\
2 & 1
\end{vmatrix} = 1 \cdot 0 = 0$$

$$D_{23} = (-1)^{2+3} \cdot \begin{vmatrix}
-4 & 2 \\
2 & 1
\end{vmatrix} = -1 \cdot (-8) = 8$$

$$D_{11} = (-1)^{1+1} \cdot \begin{vmatrix}
2 & 3 \\
1 & 0
\end{vmatrix} = 1 \cdot (-3) = -3$$

$$D_{22} = (-1)^{2+2} \cdot \begin{vmatrix}
-4 & 1 \\
2 & 0
\end{vmatrix} = 1 \cdot (-2) = -2$$

For our matrix $A$ we can compute 9 cofactors.
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**Theorem 4.5.** For square matrices $A$ of any order $n$, using the Laplace expansion, the determinant can be computed by formula

$$\det A = a_{i1}D_{i1} + a_{i2}D_{i2} + \ldots + a_{in}D_{in},$$

or

$$\det A = a_{1j}D_{1j} + a_{2j}D_{2j} + \ldots + a_{nj}D_{nj}.$$ 

**Example 4.26.** Use the Laplace expansion to find the determinant of matrix $A = \begin{bmatrix} -4 & 2 & 1 \\ 4 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$.

**Solution.** Expanding down by the second row to compute $\det A$.

$$\det A = 4 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot (-1)^{2+2} \begin{vmatrix} -4 & 1 \\ 2 & 0 \end{vmatrix} + 3 \cdot (-1)^{2+3} \begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} =$$

$$= -4 \cdot (-1) + 2 \cdot (-2) + (-3) \cdot (-8) = 24$$

Of course we obtain the same result using expanding for instance, by $3$–th column. This yields:

$$\det A = 1 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} + 3 \cdot (-1)^{2+3} \begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot (-1)^{3+3} \begin{vmatrix} -4 & 2 \\ 4 & 2 \end{vmatrix} =$$

$$= 1 \cdot 0 + (-3) \cdot (-8) + 0 = 24$$

Sometimes we can use the Laplace expansion a few times.

**Example 4.27.** Find the determinant of the matrix

$$A = \begin{bmatrix} -3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 & 0 \\ 7 & 2 & 3 & 0 & 0 \\ 4 & -1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 18 & 3 \end{bmatrix}$$

**Solution.** Let us notice, that the last column has most of zeros (which means one less set of calculations). So expanding down the last column the first round of the Laplace expansion
gives
\[
\det A = 3 \cdot (-1)^{5+5} \begin{vmatrix}
-3 & 1 & 1 \\
2 & 0 & 3 \\
7 & 2 & 3 \\
4 & -1 & 3 \\
\end{vmatrix}
\]
A second round of the Laplace expansion is expanding down the last column and so this gives
\[
= 3 \begin{vmatrix}
-3 & 1 & 1 \\
7 & 2 & 3 \\
4 & -1 & 3 \\
\end{vmatrix} + 2 \cdot (-1)^{4+4} \begin{vmatrix}
3 & 1 & 1 \\
2 & 0 & 3 \\
7 & 2 & 3 \\
\end{vmatrix} = 3[(-51) + 2 \cdot 37] = 3 \cdot 23 = 69
\]

### 4.4 Inverse Matrix

Consider the square matrix \( A \) of order \( n \).

**Definition 4.15.** Matrix inversion is the process of finding the matrix, which we denoted by \( A^{-1} \), that satisfies
\[
AA^{-1} = A^{-1}A = I_n
\]
The matrix \( A^{-1} \) is called the inverse matrix and the matrix \( A \) is invertible.

Consider the square matrix \( A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \) then the matrix \( A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \) is the inverse matrix of \( A \).

Let check it
\[
AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
and
\[
A^{-1}A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
So the matrix \( A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \) is invertible.

**Definition 4.16.** The square matrix \( A \) is called singular, if \( \det A = 0 \), if \( \det A \neq 0 \), then \( A \) is called non singular.
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**Theorem 4.6.** The square matrix $A$ is invertible if and only if $A$ is non singular.

If matrix $A = [a_{ij}]$ of order $n$ is non singular, then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n1} & D_{n2} & \cdots & D_{nn} \end{bmatrix}^T,$$

where $D_{ij}$ are $a_{ij}$ cofactors of $A$.

**Example 4.28.** Find the inverse matrix $A^{-1}$ for matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{bmatrix}$.

**Solution.** First, we must designate the determinant.

$$\det A = \begin{vmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{vmatrix} = -18 - 84 + 100 - (-90 - 16 + 105) = -2 - (-1) = -1$$

So our matrix is non singular and we can find the inverse matrix. The next step is to find $D_{ij}$.

$$D_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix} = -1,$$

$$D_{12} = (-1)^{1+2} \begin{vmatrix} 6 & 4 \\ 5 & -3 \end{vmatrix} = 38,$$

$$D_{13} = (-1)^{1+3} \begin{vmatrix} 6 & 3 \\ 5 & -2 \end{vmatrix} = -27,$$

$$D_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 7 \\ -2 & -3 \end{vmatrix} = 1,$$

$$D_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41,$$
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\[
D_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = 29,
\]

\[
D_{31} = (-1)^{3+1} \begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix} = -1,
\]

\[
D_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 7 \\ 6 & 4 \end{vmatrix} = 34,
\]

\[
D_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = -24.
\]

By above theorem, we can compute

\[
A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 38 & -27 \\ 1 & -41 & 29 \\ -1 & 34 & -24 \end{bmatrix}^T = -1 \begin{bmatrix} -1 & 1 & -1 \\ 38 & -41 & 34 \\ -27 & 29 & -24 \end{bmatrix} =
\]

\[
= \begin{bmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{bmatrix}
\]

**Theorem 4.7.** Let \(A\) and \(B\) are non singular matrices of the same order and let \(\alpha \neq 0\).

Then matrices \(A^{-1}, A^T, AB, \alpha A\) and \(A^n (n \in \mathbb{N})\) are invertible and

1. \(\det (A^{-1}) = (\det A)^{-1}\)
2. \((A^{-1})^{-1} = A\)
3. \((A^T)^{-1} = (A^{-1})^T\)
4. \((AB)^{-1} = B^{-1}A^{-1}\)
5. \((\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}\)
6. \((A^n)^{-1} = (A^{-1})^n\)

Consider \(A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\), then \(A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\) and

\[
\det A = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2
\]

\[
\det (A^{-1}) = \det \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = -2 \cdot (-\frac{1}{2}) - \frac{3}{2} \cdot 1 = 1 - \frac{3}{2} = -\frac{1}{2}
\]

Therefore

\[
\det (A^{-1}) = \frac{1}{\det A}
\]

The concept of an inverse matrix is important because it allows us to solve equations.

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**Theorem 4.8.** Consider the equation

\[ A \cdot X \cdot B = C \]

where \( A, X, B, C \) are invertible matrices of order \( n \). Then the matrix \( X \) is of the form

\[ X = A^{-1} \cdot C \cdot B^{-1} \]

**Example 4.29.** Solve the equation

\[
\begin{bmatrix}
  1 & 2 \\
  3 & 4 \\
\end{bmatrix}
\cdot 
\begin{bmatrix}
  1 & 4 \\
  5 & 1 \\
\end{bmatrix}
\cdot 
\begin{bmatrix}
  67 & 78 \\
  137 & 168 \\
\end{bmatrix}
\]

**Solution.** For \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) the inverse function is of the form \( A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \).

For \( B = \begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix} \) the inverse function is of the form \( B^{-1} = \begin{bmatrix} -\frac{1}{19} & \frac{4}{19} \\ \frac{5}{19} & -\frac{1}{19} \end{bmatrix} \). Therefore, we can solve the equation

\[ X = A^{-1} \cdot C \cdot B^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \cdot 
\begin{bmatrix}
  67 & 78 \\
  137 & 168 \\
\end{bmatrix} \cdot 
\begin{bmatrix}
  -\frac{1}{19} & \frac{4}{19} \\
  \frac{5}{19} & -\frac{1}{19} \\
\end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 32 & 33 \end{bmatrix} \]

we must do some calculations

\[
\begin{bmatrix}
  -2 & 1 \\
  \frac{3}{2} & -\frac{1}{2} \\
\end{bmatrix} \cdot 
\begin{bmatrix}
  67 & 78 \\
  137 & 168 \\
\end{bmatrix} = \begin{bmatrix}
  -2 \cdot 67 + 1 \cdot 137 & -2 \cdot 78 + 1 \cdot 168 \\
  \frac{3}{2} \cdot 67 + (\frac{-1}{2}) \cdot 137 & \frac{3}{2} \cdot 78 + (\frac{-1}{2}) \cdot 168 \\
\end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 32 & 33 \end{bmatrix}
\]

and

\[
\begin{bmatrix}
  3 & 12 \\
  32 & 33 \\
\end{bmatrix} \cdot 
\begin{bmatrix}
  -\frac{1}{19} & \frac{4}{19} \\
  \frac{5}{19} & -\frac{1}{19} \\
\end{bmatrix} = \begin{bmatrix}
  3 \cdot (-\frac{1}{19}) + 12 \cdot \frac{5}{19} & 3 \cdot \frac{4}{19} + 12 \cdot (-\frac{1}{19}) \\
  32 \cdot (-\frac{1}{19}) + 33 \cdot \frac{5}{19} & 32 \cdot \frac{4}{19} + 33 \cdot (-\frac{1}{19}) \\
\end{bmatrix} = 
\begin{bmatrix}
  \frac{57}{19} & 0 \\
  \frac{133}{19} & \frac{95}{19} \\
\end{bmatrix} = \begin{bmatrix}
  3 & 0 \\
  7 & 5 \\
\end{bmatrix}
\]

Therefore finally

\[ X = A^{-1} \cdot C \cdot B^{-1} = \begin{bmatrix} 3 & 0 \\ 7 & 5 \end{bmatrix} \]
4.5 Systems of linear equations

**Definition 4.17.** The system of \( m \) linear equations with \( n \) unknowns \( x_1, x_2, \ldots, x_n \), is of the form

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  \vdots & \vdots \ddots \vdots \\\n  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

(4.1)

where the coefficients \( a_{ij}, b_i \in \mathbb{R} \) for \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, n\} \).

A solution of a linear system is an assignment of values to the variables \( x_1, x_2, \ldots, x_n \), such that each of the equations is satisfied. The linear system of equations has either no solution or a single unique solution or an infinite number of solutions.

**Example 4.30.** The system

\[
\begin{align*}
  x_1 + 2x_2 + x_3 + x_4 - x_5 - x_6 &= 1 \\
  x_1 - 4x_2 + x_3 &= 1 \\
  3x_1 - 2x_2 + \frac{1}{5}x_3 + 2x_4 - 3x_5 + 10x_6 &= 7 \\
  4x_1 + 2x_2 - x_3 + 5x_4 + 2x_5 - 6x_6 &= 8 \\
  x_1 + x_3 + x_4 + x_5 + x_6 &= 6 \\
  x_1 + 6x_2 - 7x_3 + 12x_4 + x_5 + 3x_6 &= 0
\end{align*}
\]

has exactly one solution of the form

\[ x_1 = 3, \ x_2 = 0.5, \ x_3 = 0, \ x_4 = -1, \ x_5 = 3, \ x_6 = 1 \]

The system of \( m \) linear equations with \( n \) unknowns can be written in matrix format as

\[ \mathbf{Ax} = \mathbf{b}, \]

where \( \mathbf{A} \) is the \( m \times n \) matrix of coefficients.

\[
\mathbf{A} = 
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}, \quad
\mathbf{x} = 
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}, \quad
\mathbf{b} = 
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}.
\]
Example 4.31. Write the system in matrix format

\[
\begin{aligned}
&x_1 + 2x_2 + x_3 + x_4 - x_5 - x_6 = -1 \\
x_1 - 4x_2 + x_3 = 1 \\
3x_1 - 2x_2 + \frac{1}{7}x_3 + 2x_4 - 3x_5 + 10x_6 = 7 \\
4x_1 + 2x_2 - x_3 + 5x_4 + 2x_5 - 6x_6 = 8 \\
x_1 + x_3 + x_4 + x_5 + x_6 = 6 \\
x_1 + 6x_2 - 7x_3 + 12x_4 + x_5 + 3x_6 = 0
\end{aligned}
\]

Solution. So, we have

\[
A = \begin{bmatrix}
1 & 2 & 1 & 1 & -1 & -1 \\
1 & -4 & 1 & 0 & 0 & 0 \\
3 & -2 & \frac{1}{7} & 2 & -3 & 10 \\
4 & 2 & -1 & 5 & 2 & -6 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 6 & -7 & 12 & 1 & 3
\end{bmatrix}, \quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}, \quad b = \begin{bmatrix}
-1 \\
1 \\
7 \\
8 \\
6 \\
0
\end{bmatrix}
\]

4.5.1 Cramer’s Rule

The Cramer’s rule is a method of using matrices for solving system of linear equations. It is very useful in economics.

Definition 4.18. The system of linear equations

\[
Ax = b
\]  \hspace{1cm} (4.2)

where the matrix A is non singular we called the Cramer’s system.

Theorem 4.9. The Cramer’s system has exactly one solution and it is given by formula

\[
x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \ldots, x_n = \frac{\det A_n}{\det A},
\]

where \(A_j j \in \{1, 2, \ldots, n\}\) denotes a matrix which can be found by substituting the vector of constant values \(b\) for the \(j\)–th column of matrix \(A\).
Example 4.32. Find the solution of the following system

\[
\begin{align*}
-x_1 + x_2 + x_3 &= 3 \\
2x_1 + x_2 - x_3 &= 0 \\
x_1 + 3x_2 + 2x_3 &= 5
\end{align*}
\]

Solution. First of all, we must compute determinant of the matrix of the coefficients

\[
\det A = \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix} = 5
\]

So our system is the Cramer’s system, therefore

\[
\begin{align*}
\det A_1 &= \begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 5 & 3 & 2 \end{vmatrix} = 5, \\
\det A_2 &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 5 & 2 \end{vmatrix} = 0, \\
\det A_3 &= \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 5 \end{vmatrix} = 10
\end{align*}
\]

Finally, the solution is of the form

\[
x_1 = \frac{\det A_1}{\det A} = \frac{5}{5} = 1, \quad x_2 = \frac{\det A_2}{\det A} = \frac{0}{5} = 0, \quad x_3 = \frac{\det A_3}{\det A} = \frac{10}{5} = 2
\]

Theorem 4.10. The solution of the Cramer’s system \( Ax = b \) is of the form

\[
x = A^{-1}b
\]

Example 4.33. Find the solution

\[
\begin{align*}
x_1 + x_2 + x_3 &= 3 \\
2x_1 + x_2 - x_3 &= 0 \\
x_1 + 3x_2 + 2x_3 &= 5
\end{align*}
\]

Solution. We can write the system in matrix format

\[
\leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}
\]
Then the inverse matrix of the matrix of the coefficients is of the form \( A^{-1} = \begin{bmatrix} 1 & \frac{1}{5} & -\frac{2}{5} \\ -1 & \frac{1}{5} & \frac{3}{5} \\ 1 & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix} \).

So the solution can be found by the formula

\[
\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{5} & -\frac{2}{5} \\ -1 & \frac{1}{5} & \frac{3}{5} \\ 1 & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 - 2 \\ -3 + 3 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}
\]

Therefore \( x_1 = 1, x_2 = 0, x_3 = 2 \).
4.6 Review Exercises

Exercise 4.1. Evaluate and compute

a) \[ \sum_{i=1}^{4} (4i + 2) \]  
b) \[ \sum_{k=1}^{10} (k - 5) \]  
c) \[ \sum_{i=2}^{7} (2i - 3) \]  
d) \[ \sum_{i=1}^{9} (i + 2) - \sum_{i=1}^{2} \]

e) \[ \sum_{i=8}^{10} (5i - 5) \]  
f) \[ \sum_{k=5}^{12} (2k + 7) \]  
g) \[ \sum_{k=2}^{6} 2(2^k + 1) \]  
h) \[ \sum_{i=1}^{3} \]

i) \[ \sum_{i=1}^{8} (i^4 - 3) - \sum_{i=1}^{8} i^4 \]  
j) \[ \sum_{k=0}^{5} (2k + 1) + \sum_{k=0}^{3} k \]  
k) \[ \sum_{k=3}^{6} \frac{k}{k+1} \]  
l) \[ \sum_{i=1}^{4} 3^i \]

Exercise 4.2. Compute:

a) \[ \sum_{i=1}^{5} \sum_{k=1}^{4} (k + 1) \]  
b) \[ \sum_{i=1}^{5} \sum_{k=1}^{4} (i + 2) \]  
c) \[ \sum_{i=1}^{2} \sum_{k=1}^{3} i \cdot k \]  
d) \[ \sum_{i=1}^{2} \sum_{k=1}^{3} (2k + i + 1) \]

Exercise 4.3. Let \( x_{ik} \) represents the profit generated in the \( i \)th \( (1 \leq i \leq 10) \) enterprise in the month of number \( k \) \( (1 \leq k \leq 12) \).

1. What is the meaning of the following sum:

a) \[ \sum_{k=1}^{12} x_{1k} \]  
b) \[ \sum_{k=7}^{9} x_{3k} \]  
c) \[ \sum_{i=1}^{7} x_{i5} \]  
d) \[ \sum_{i=1}^{10} \sum_{k=1}^{12} x_{ik} \]  
e) \[ \sum_{i=6}^{12} \sum_{k=4}^{6} x_{ik} \]  
f) \[ \sum_{k=1}^{12} \sum_{i=1}^{10} x_{ik} \]  
g) \[ \sum_{i=1}^{10} \sum_{k=1}^{12} x_{ik} \]  
h) \[ \sum_{i=1}^{10} \sum_{k=10}^{12} x_{ik} \]

2. Sign symbolic value of enterprises profits in the month of April.

3. Sign average profit of enterprises in the first quarter.

Exercise 4.4. Compute the matrices \( A + B, 2A, 2A - B \)

a) \[ A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \] \[ B = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \]  
b) \[ A = \begin{bmatrix} 0 & 7 & 1 \\ 1 & 0 & 4 \\ -2 & 1 & 3 \end{bmatrix} \] \[ B = \begin{bmatrix} 0 & 7 & 4 \\ -2 & 1 & 3 \end{bmatrix} \]

c) \[ A = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 4 & -2 \end{bmatrix} \] \[ B = \begin{bmatrix} 2 & 2 & \sqrt{2} \\ 1 & 5 & -\sqrt{3} \end{bmatrix} \]  
d) \[ A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \] \[ B = \begin{bmatrix} 2 & 7 \\ 0 & 3 \end{bmatrix} \]

Exercise 4.5. Find the product matrix \( A \cdot B \), where

a) \[ A = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 4 & -2 \end{bmatrix} \] \[ B = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \]  
b) \[ A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \] \[ B = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \]

c) \[ A = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \] \[ B = \begin{bmatrix} -2 & 0 \end{bmatrix} \]  
d) \[ A = \begin{bmatrix} 1 & 7 \end{bmatrix} \] \[ B = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \]
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Exercise 4.6. Solve the matrix equation:

\[ a) \begin{bmatrix} 1 & -4 & -2 \\ 3 & 7 & 0 \end{bmatrix} + X = \begin{bmatrix} 3 & -5 & 2 \\ 1 & -11 & 3 \end{bmatrix} \]
\[ b) \begin{bmatrix} -1 & 0 & 3 \\ 1 & 1 & 4 \\ 1 & -2 & 5 \end{bmatrix} + X = 10I \]
\[ c) \begin{bmatrix} 4 & -4 & 4 \\ 8 & 0 & -4 \end{bmatrix} - 2X = X + \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 4 & -1 \end{bmatrix}^T \]
\[ d) \begin{bmatrix} 2 & 2 & 3 \\ 5 & 3 & 4 \end{bmatrix} - X = I \]

Exercise 4.7. Consider matrices
\[ A = \begin{bmatrix} 0 & 1 \\ 10 & 2 \end{bmatrix}, \quad i \ B = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \]. Verify the following properties:

1. \((A^T)^T = A\)
2. \((A + B)^T = A^T + B^T\)
3. \((AB)^T = B^T A^T\)

Exercise 4.8. Consider the matrix
\[ A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix} \] and \( B = [b_{ij}] \), where \( b_{11} = 3, b_{12} = 1, b_{31} = 0, b_{32} = -2 \) and \( b_{21} \) and \( b_{22} \) are solutions of the equation \( 2x^2 + x - 1 \) and \( b_{21} < b_{22} \).

1. compute the matrix \( C = B + A^T \)
2. compute the matrix \( D = BA \)
3. compute the matrix \( E = B^T A^T + 8I \)

Exercise 4.9. Consider \( 2 \times 2 \) matrices
\[ A = \begin{cases} i - j & \text{dla } i > j \\ 2 & \text{dla } i \leq j \end{cases} \] and \( B = \begin{cases} 2i + j & \text{dla } i \neq j \\ 1 & \text{dla } i = j \end{cases} \)

1. compute the matrix \( C = AB \)
2. compute the matrix \( D = B^T A^T \)
3. compute the matrix \( E = 4A - 2B + 8I \)
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Exercise 4.10. Compute determinants

a) \[
\begin{vmatrix}
7 & 0 \\
1 & -1
\end{vmatrix}
\]
b) \[
\begin{vmatrix}
1 & 3 & -2 \\
2 & 4 & 5 \\
-1 & 0 & -2
\end{vmatrix}
\]
c) \[
\begin{vmatrix}
1 & 0 & 1 & 7 \\
0 & -3 & 2 & 2 \\
0 & 0 & 4 & -4 \\
0 & 0 & 0 & 5
\end{vmatrix}
\]
d) \[
\begin{vmatrix}
2 & 1 & 1 \\
2 & 1 & 3
\end{vmatrix}
\]
e) \[
\begin{vmatrix}
-1 & 2 & 1 \\
1 & 5 & 1 \\
3 & -8 & 1
\end{vmatrix}
\]
f) \[
\begin{vmatrix}
4 & -7 & 2 \\
1 & 1 & 4 \\
1 & 8 & 7
\end{vmatrix}
\]
g) \[
\begin{vmatrix}
174 & 1 & 2 & \frac{1}{2} \\
1 & 0 & 1 & -7 \\
-1 & 0 & 5 & 2 \\
4 & 0 & 2 & 1
\end{vmatrix}
\]
h) \[
\begin{vmatrix}
1 & 2 & 1 \\
1 & 0 & 0 \\
2 & 1 & 2 \\
-1 & 1 & 1
\end{vmatrix}
\]

Exercise 4.11. Find the inverse matrix (if it exists)

a) \[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]
b) \[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 0 & 1 \\
2 & 1 & 3
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
2 & 2 & 3 \\
1 & -1 & 0 \\
-1 & 2 & 1
\end{bmatrix}
\]

Exercise 4.12. Verify the following properties (A^{-1})^{-1} = A, (A^T)^{-1} = (A^{-1})^T, (2A)^{-1} = \frac{1}{2}A^{-1} and det A^{-1} = \frac{1}{\det A}, if

a) \[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 4
\end{bmatrix}
\]
b) \[
A = \begin{bmatrix}
2 & -1 & 0 \\
0 & 1 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\]

Exercise 4.13. Solve equations

a) \[
\begin{bmatrix}
2 & 2 \\
1 & 0
\end{bmatrix}
\cdot X = \begin{bmatrix}
1 & 4 \\
2 & 1
\end{bmatrix}
\]
b) \[
X \cdot \begin{bmatrix}
2 & 2 \\
1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 4 \\
2 & 1
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
1 & 2 \\
1 & 0
\end{bmatrix}
\cdot X \cdot \begin{bmatrix}
2 & 3 \\
2 & 1
\end{bmatrix} = \begin{bmatrix}
3 & 2 \\
0 & 1
\end{bmatrix}
\]
d) \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\cdot X \cdot \begin{bmatrix}
-2 & 1 \\
3 & 6
\end{bmatrix} = \begin{bmatrix}
-12 & 0 \\
0 & 8
\end{bmatrix}
\]

Exercise 4.14. Find the solution of the linear system
Chapter 4. Introduction to algebra

a) \[
\begin{align*}
    x_1 - 3x_2 + 5x_3 &= -4 \\
    2x_1 + 5x_2 - x_3 &= 3 \\
    -x_1 - x_2 + 3x_3 &= -4
\end{align*}
\]
b) \[
\begin{align*}
    -x_1 + 2x_2 - x_3 &= 2 \\
    3x_1 - x_2 + x_3 &= \frac{1}{2} \\
    2x_1 + 8x_2 - 3x_3 &= 12
\end{align*}
\]
c) \[
\begin{align*}
    5x_1 - 3x_2 + 7x_3 &= 0 \\
    -4x_1 + x_2 - 5x_3 &= 0 \\
    x_1 - x_2 + x_3 &= 0
\end{align*}
\]
d) \[
\begin{align*}
    x_2 - 3x_3 + 4x_4 &= 0 \\
    x_1 - 2x_3 &= 0 \\
    3x_1 + 2x_2 - 5x_4 &= 2 \\
    4x_1 - 5x_3 &= 0
\end{align*}
\]
e) \[
\begin{align*}
    x_1 - 2x_2 &= -2 \\
    2x_2 + x_3 &= 1 \\
    x_1 - x_3 &= 1
\end{align*}
\]
f) \[
\begin{align*}
    x_1 + 3x_2 - x_3 &= 8 \\
    x_1 + x_2 - 3x_3 &= 2 \\
    2x_2 + x_3 &= 5
\end{align*}
\]
g) \[
\begin{align*}
    x_1 + x_2 + x_3 &= 0 \\
    2x_1 - x_2 - x_3 &= -3 \\
    x_1 - x_2 + x_3 &= 0
\end{align*}
\]
h) \[
\begin{align*}
    \begin{bmatrix}
        1 & 1 & -1 \\
        1 & -3 & 2 \\
        -1 & 2 & -1
    \end{bmatrix}
    \begin{bmatrix}
        x_1 \\
        x_2 \\
        x_3
    \end{bmatrix}
    &= \begin{bmatrix}
        -2 \\
        0 \\
        1
    \end{bmatrix}
\end{align*}
\]
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Answers

Exercise 4.1 a) 48 b) 5 c) 36 d) 45 e) 770 f) 560 g) 128 h) 300 i) \(-24\) j) 51  k) \(\frac{31}{20}\)  l) 120

Exercise 4.2 a) 70 b) 100 c) 18 d) 36

Exercise 4.3 1. a) annual profit generated in the enterprise No. 1
b) profit generated in the enterprise No. 3 from July to September
c) profit generated by enterprises in May, with numbers from 1 to 7
d) annual profit generated in all enterprises
e) profit generated by enterprises with numbers 6 and 7 in the second quarter
f) average monthly profit in all enterprises
g) average annual profit
h) profit generated by enterprises in the fourth quarter

2. \(\sum_{i=1}^{10} x_{i4}\)

3. \(\frac{1}{10} \sum_{i=1}^{10} \sum_{k=1}^{k} x_{ik}\)

Exercise 4.4 a) \(A + B = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix},\) \(2A = \begin{bmatrix} 6 & 2 \\ 0 & -4 \end{bmatrix},\) \(2A - B = \begin{bmatrix} 8 & -2 \\ -3 & -9 \end{bmatrix}\), b) \(A + B = \begin{bmatrix} -2 & 8 & 4 \\ 1 & 7 & 8 \end{bmatrix},\) \(2A = \begin{bmatrix} 0 & 14 & 2 \\ 2 & 0 & 8 \end{bmatrix},\) \(2A - B = \begin{bmatrix} 2 & 13 & -1 \\ 2 & -7 & 4 \end{bmatrix}\), c) \(2A = \begin{bmatrix} -1 & \frac{7}{2} \\ -1 & 5 \end{bmatrix},\) \(2A - B = \begin{bmatrix} -4 & 1 \\ -4 & 6 \end{bmatrix}\), d) \(A + B = \begin{bmatrix} 3 & -1 & \sqrt{2} \\ 4 & 9 & -\sqrt{3} - 2 \end{bmatrix},\) \(2A = \begin{bmatrix} 2 & -6 & 0 \\ 6 & 8 & -4 \end{bmatrix},\) \(2A - B = \begin{bmatrix} 0 & -8 & -\sqrt{2} \\ 5 & 3 & \sqrt{3} - 4 \end{bmatrix}\)

Exercise 4.5 a) \(A \cdot B = \begin{bmatrix} -2 \\ -8 \end{bmatrix}\), b) \(A \cdot B = \begin{bmatrix} -3 & 17 \\ -6 & -10 \end{bmatrix}\), c) \(A \cdot B = \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}\), d) \(A \cdot B = [43]\)
Exercise 4.6 a) \( X = \begin{bmatrix} 1 & 3 & 6 \\ -5 & -25 & 3 \end{bmatrix} \), b) \( X = \begin{bmatrix} 11 & 0 & -3 \\ -1 & 9 & -4 \\ -1 & 2 & 5 \end{bmatrix} \), c) \( X = \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} & -1 \\ -\frac{5}{3} & 0 & 0 \end{bmatrix} \), d)

\[
X = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 4 \\ 4 & 0 & 0 \end{bmatrix}
\]

Exercise 4.8. \( B = \begin{bmatrix} 3 & 1 \\ -1 & \frac{1}{2} \\ 0 & -2 \end{bmatrix} \), \( C = \begin{bmatrix} 3 & -1 \\ 0 & \frac{3}{2} \\ -1 & 0 \end{bmatrix} \), \( D = \begin{bmatrix} -2 & 4 & -1 \\ -1 & -\frac{1}{2} & 2 \\ 4 & -2 & -4 \end{bmatrix} \), \( E = \begin{bmatrix} 7 & -7 \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix} \)

Exercise 4.9. \( A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \), \( C = \begin{bmatrix} 16 & 20 \\ 13 & 16 \end{bmatrix} \), \( D = \begin{bmatrix} 11 & 16 \\ 14 & 20 \end{bmatrix} \), \( E = \begin{bmatrix} 10 & 0 \\ -6 & 4 \end{bmatrix} \)

Exercise 4.11

a) \(-7\) b) \(-19\) c) \(-60\) d) \(-12\) e) \(-32\) f) \(-65\) g) \(-164\) h) 12

Exercise 4.13

a) \( X = \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & 1 \end{bmatrix} \), b) \( X = \begin{bmatrix} -1 & 3 \\ \frac{3}{2} & -1 \end{bmatrix} \), c) \( X = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{8} & \frac{7}{8} \end{bmatrix} \), d) we can not solve the equation

\[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot X \cdot \begin{bmatrix} -2 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} -12 & 0 \\ 0 & 8 \end{bmatrix}, \text{because there is no inverse matrix to the matrix}
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

\( \text{because det} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 \)

Exercise 4.14

a) \( x_1 = 1, x_2 = 0, x_3 = -1 \), b) \( x_1 = 1, x_2 = \frac{1}{2}, x_3 = -2 \), c) \( x_1 = 0, x_2 = 0, x_3 = 0 \), d) \( x_1 = 0, x_2 = \frac{8}{15}, x_3 = 0, x_4 = -\frac{2}{15} \), e) \( x_1 = 0, x_2 = 1, x_3 = -1 \), f) \( x_1 = 3, x_2 = 2, x_3 = 1 \), g) \( x_1 = -1, x_2 = 0, x_3 = 1 \), h) \( x_1 = -1, x_2 = 1, x_3 = 2 \).
Chapter 5

Graph theory

5.1 Basic Graph Theory

5.1.1 Undirected and directed graph

Definition 5.1. The undirected graph (or shortly graph) is a pair:

\[ G = (V, E) \]

where \( V \) is a nonempty set and we called them the set of vertices of \( G \) and \( E = \{ \{u, v\} : u, v \in V \} \) is a set of edges \( G \).

Example 5.1. Consider the undirected graph \( G = (V, E) \), where \( V = \{a, b, c, d, e\} \), \( E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, e\}\} \).

We usually we use a graphical representation of a graph. The vertices of a graph \( G \) are represented by points and the edges by lines connecting the end points.

\[
\begin{array}{c}
\text{e} \\
\text{b} \\
\text{a} \\
\text{c} \\
\text{d}
\end{array}
\]

Remark 5.1. Let \( \{p, q\} \in E \). We sometimes use a notation \( m = \{p, q\} \), when the edge \( m \) connect the vertex \( p \) and the vertex \( q \). Moreover in this case \( p \) and \( q \) we called the end points of
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$m$ and we can say, that $p$ and $q$ are incident with $m$ and that $p$ and $q$ are adjacent or neighbors of each other.

**Remark 5.2.** In many applications we use a spacial types of edges: multiedges and loops. Multiedges are edges which connect we same vertices and a loop connects the vertex with itself.

On this picture edges $m$ and $l$ are multiedges and $d$ and $b$ are loops. A graph with multiedges and loops are called multigraph.

Often, we associate weights to edges of the graph. These weights can represent cost, profit or loss, length, capacity etc. of given connection.

**Definition 5.2.** The weight is a mapping from set of edges to real numbers $w : E \mapsto R$.

**Remark 5.3.** The graph with weight function we sometimes called the network and denoted by $G = \langle V, E, w \rangle$.

**Example 5.2.** Now let us consider the graph $G = \langle V, E, w \rangle$, where $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{e, b\}, \{d, b\}, \{c, d\}, \{d, e\}\}$ with the weight function

<table>
<thead>
<tr>
<th>krawędź $e \in E$</th>
<th>{a,b}</th>
<th>{a,c}</th>
<th>{a,e}</th>
<th>{d,b}</th>
<th>{e,b}</th>
<th>{d,e}</th>
<th>{d,c}</th>
</tr>
</thead>
<tbody>
<tr>
<td>waga krawędzi $w(e)$</td>
<td>-9</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>12</td>
<td>8</td>
<td>0.7</td>
</tr>
</tbody>
</table>

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Example 5.3. Consider schools of languages \( SH = \{\text{yes, ok, maybe, ling}\} \) and students \( S = \{\text{Ania, Ela, Tomek, Edek}\} \) the set of potential students. Every student is interested in some school \( SH_i \subset SH \). This situation can be easy modeling on the graph \( G = \langle SH \cup S, E \rangle \).

Definition 5.3. The directed graph (digraph) is a pair:

\[
G = \langle V, E \rangle
\]

where \( V \) is a nonempty set of vertices and \( E = \{(u, v) : u, v \in V\} \) is a set of directed edges.

In digraphs, \( E \) is collection of ordered pairs. If \((p, q) \in E\), \( p \) is the head of edge and \( q \) is the tail of edge.

Example 5.4. Consider the directed graph \( G = \langle V, E \rangle \), where \( V = \{a, b, c, d, e\} \), \( E = \{(a, b), (a, c), (a, e), (e, b), (d, b), (c, d), (d, e)\} \).

Remark 5.4. In graph theory we can also consider the digraph \( \langle V, E, w \rangle \) with the weight function.
**Example 5.5.** A dispatcher for a cab company can communicate two way with each cab and one way with a customer. A digraph of this communication model might look like picture below.

![Diagram of communication model](image)

**Remark 5.5.** Most of the definitions and properties for directed and undirected graphs are the same. Otherwise, it will be marked.

### 5.1.2 Basic properties of graphs

**Definition 5.4.** The degree of a vertex \( v \), \( \deg(v) \) is the number of edges for which that vertex is incident (in case loops we use them twice). The number of vertices of degree \( k \) we denoted by \( D_k(G) \). For every graph \( G \) we can define a sequence of degrees \( (D_0(G), D_1(G), D_2(G), \ldots) \).

A vertex \( v \) with degree 0, \( \deg(v) = 0 \) is called an isolated vertex, a vertex \( u \) with degree 1, \( \deg(u) = 1 \) is called the end vertex, or the hanging vertex, or in some special family of graphs, a leaf.

**Definition 5.5.** The degree of a graph \( G \), \( \Delta(G) \) we denote the highest degree of all vertices

\[
\Delta(G) = \max_{v \in V} \deg(v).
\]

**Example 5.6.** Consider the graph
then

1. the isolated vertices $x_5$ and $x_7$,

2. the end vertices $x_4$ and $x_6$,

3. $\deg(x_1) = 2$ and $\deg(x_2) = 5$, $\deg(x_3) = 4$ and $\deg(x_8) = 3$

4. the sequence of degrees $(2, 2, 1, 1, 1, 1)$,

5. the degree of graph $\Delta(G) = 5$.

**Definition 5.6.** The indegree of a vertex $v$, $\text{degin}(v)$ is the number of head endpoints adjacent to $v$. The outdegree of a vertex $v$, $\text{degout}(v)$ is the number of tail endpoints adjacent from $v$. The degree of a vertex $\deg(v)$ in digraph is equal

$$\deg(v) = \text{degout}(v) + \text{degin}(v)$$

**Example 5.7.** Consider the digraph
Chapter 5. Graph theory

\[ \text{degout}(x) = 3, \quad \text{degin}(x) = 2, \]
\[ \text{degout}(u) = 2, \quad \text{degin}(u) = 1, \]
\[ \text{degout}(z) = 4, \quad \text{degin}(z) = 1, \]
\[ \text{degout}(w) = 1, \quad \text{degin}(w) = 2, \]
\[ \text{degout}(y) = 1, \quad \text{degin}(y) = 5. \]

**Definition 5.7.** The degree of a vertex \( \text{deg}(x) \) in digraph is the sum

\[ \text{deg}(x) = \text{degout}(x) + \text{degin}(x) \]

**Definition 5.8.** A vertex with \( \text{degin}(v) = 0 \) is called a source and a vertex with \( \text{degout}(v) = 0 \) is called a sink.

**Example 5.8.** Consider the digraph

We have the following degrees for vertices

\[ \text{deg}(a) = \text{degout}(a) + \text{degin}(a) = 1 + 0 = 1 \]
\[ \text{deg}(b) = \text{degout}(b) + \text{degin}(b) = 1 + 2 = 3 \]
\[ \text{deg}(c) = \text{degout}(c) + \text{degin}(c) = 2 + 0 = 2 \]
\[ \text{deg}(d) = \text{degout}(d) + \text{degin}(d) = 1 + 1 = 2 \]
\[ \text{deg}(e) = \text{degout}(e) + \text{degin}(e) = 0 + 2 = 2 \]

The degree of digraph

\[ \Delta(G) = \max_{u \in V} \text{deg}(u) = 3 \]

Sources in digraph are vertices \( a \) and \( c \). The sink is the vertex \( e \).
**Theorem 5.1.** For any graph we have

\[\sum_{v \in V} \deg(v) = 2 |E|\]

**Theorem 5.2.** The number of vertices of odd degree is even.

Let us denote an edge \( e \in E \) for directed or undirected graph as \( e = pq \) and let \((e_1, \ldots, e_n)\) be a sequence of edges.

**Definition 5.9.** If there are vertices \( v_0, \ldots, v_n \) such that \( e_i = \{v_{i-1}, v_i\} \) (or \( e_i = (v_i, v_{i+1}) \)), \( i = 1, \ldots, n - 1 \), then the sequence is called a walk. If \( v_0 = v_n \) the sequence is called a closed walk.

A walk for which the edges \( e_i \) are distinct is called a trail. A trail for which all the vertices \( v_i \) are distinct, we called a path (or directed path). A closed path is called a cycle (vertices \( v_i \) are distinct, except, \( v_0 = v_n \)).

The walk \( e_1 e_2 \ldots e_n \) is the walk of length \( n \) from the vertex \( v_1 \) to the vertex \( x_{n+1} \).

**Example 5.9.** Consider the graph

- the walk \( degba \) is a closed walk.
- the walk \( degbac \) is a trail \( x_1, x_5, x_5, x_3, x_2, x_1, x_3 \) but it is not a path
- the walk \( dgba \) is closed trail and it has the following distinct vertices \( x_1, x_5, x_3, x_2 \) so it is a cycle.

The directed and undirected graphs are related.
**Definition 5.10.** Let $G = (V, E)$ be a directed multigraph. Replacing each directed edge $(u,v)$ by an undirected edge $\{u,v\}$, we obtain the underlying multigraph $|G|$. Conversely, let $G = (V, E)$ be a undirected multigraph. Any directed multigraph $H$, such that $|H| = G$ is called an orientation of $G$.

![Diagram of a directed multigraph and its underlying multigraph]

**Definition 5.11.** Let $G = (V, E)$ be a digraph. A sequence of edges $(e_1, ..., e_n)$ is called a trail if the corresponding sequence of edges in $|G|$ is a trail. In the same way we can define walks, paths, closed trails and cycles.

Let us noticed, that if $(v_1, ..., v_{n+1})$ is the corresponding sequence of vertices, of trail in directed graph, then or $\{v_i, v_{i+1}\}$ or $\{v_{i+1}, v_i\}$ is an edge in graph $G$.

**Definition 5.12.** The edge $\{v_i, v_{i+1}\}$ we called a forward edge and $\{v_{i+1}, v_i\}$ a backward edge. If a trail consists of forward edges only, it is called a directed trail(walks, closed trails, etc.)

**Example 5.10.** In the above digraph $G$ we can find the trail, which is given by the sequence $(v_2, v_1, v_3, v_4)$. Therefore

- the forward edges are $(v_1, v_3)$ and $(v_3, v_4)$,
- the backward edge is $(v_2, v_1)$.

**Definition 5.13.** The graph without cycles we called acyclic.

**Definition 5.14.** The graph $G$ is called connected if for any two vertices $u$ and $v$ there exists a walk with start vertex $u$ and end vertex $v$. 

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Definition 5.15. The connected and acyclic graph we called a tree.

Of course the graph e) is not a tree. Now some properties of trees.

Theorem 5.3. Let $T = \langle V, E \rangle$ be a graph with $n$ vertices $|V| = n$. The following conditions are equivalent

1. $T$ is a tree,
2. $T$ is acyclic and has $|E| = n - 1$ edges,
3. $T$ is connected and has $|E| = n - 1$ edges,
4. $T$ is connected, but it is no longer connected after removing any edge,
5. every two vertex are connected by an exactly one path,
6. $T$ is acyclic, but adding of any new edge, yields a cycle.
**Definition 5.16.** A complete graph $K_n$ is a graph with $n$ vertices and all the pairs of vertices are adjacent to each other.

In $K_n$, $\text{deg}(v) = n - 1$.

**Theorem 5.4.** A complete graph $K_n$ has

$$|E_{K_n}| = \frac{n(n - 1)}{2}$$

edges.

**Definition 5.17.** A graph $H = (V_H, E_H)$ is subgraph of $G = (V_G, E_G)$ when $V_H \subset V_G$ and $E_H \subset E_G$.

**Example 5.11.** Consider graphs

$G_1$ and $G_2$ are subgraphs of $G$

$G_1 \subset G$, $G_2 \subset G$
5.1.3 Euler cycle

Definition 5.18. An Eulerian trail in a multigraph $G$ is a trail which contains each edge of $G$ exactly one. If the Eulerian trail is closed, then it is called an Euler cycle. If a multigraph contains an Euler cycle, then is called a Eulerian graph.

Remark 5.6. The name an Euler cycle has a historical background, because it is easy to show an Eulerian cycle which is not a cycle in the sense of previous definition.

Example 5.12. Consider graphs

in case a the multigraph has an Eulerian cycle,

in case b the multigraph has an Eulerian trail,

in case c the multigraph hasn’t an Eulerian cycle or an Eulerian trail.

Theorem 5.5. Let $G$ be a connected multigraph. $G$ is Eulerian if and only if each vertex of $G$ has even degree.

Theorem 5.6. Let $G$ be a connected multigraph. $G$ has an Eulerian trail if and only if there are exactly two vertices of odd degree in $G$.

Problem of the Seven Bridges of Königsberg

In the town Królewiec, there are two islands $A$ and $B$ on the river Pregola. They are connected each other and with shores $C$ and $D$ by seven bridges. You must set off from any part of the city land: $A, B, C$ or $D$, go through each of the bridges exactly once and return to the starting point (without passage across the river). In 1736 this problem had been solved by the Swiss mathematician Leonhard Euler (1707-1783). He built a graph representing this problem.
Leonard Euler answered the important question for citizens "has the resulting graph a closed walk, which contains all edges only once?" Of course the answer was negative.

5.1.4 Chinese postman’s problem

This problem has been formulated by the Chinese mathematician Kwan Mei Ku. Postman leaving the post office has to walk around all the streets in his area and return to the building. In the language of graph theory, in the connected, weight graph you have to find a closed trail with a minimum number of edges and minimum weight.

- If the graph is an Eulerian graph, the solution of the problem is uniformly and it is any Euler cycle.

- If the graph has an Eulerian trail, then solution to the problem is the Euler trail and the shortest return path to the starting point.

- In the other cases, solving the problem of mail delivery involves to designate certain edges that need to be moved several times. In other words, we complement the picture of the graph by multiple edges, making it the Euler graph.

5.1.5 Hamiltonian cycles

Definition 5.19. A Hamiltonian path in graph is the path that visits each vertex exactly once. A Hamiltonian cycle is a closed Hamiltonian path. If a graph contains a Hamiltonian cycle, then is called a Hamiltonian graph.
Example 5.13. Consider graphs

\[ \text{a) } \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (y); \\
\end{tikzpicture}
\end{array} \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (z); \\
\end{tikzpicture}
\end{array} \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (w); \\
\end{tikzpicture}
\end{array} 
\]

in case a the graph has a Hamilton cycle,

in case b the graph has an Hamilton path,

in case c the graph hasn’t a Hamilton cycle or a Hamilton path.

Theorem 5.7. Every complete graph \( K_n \) has a Hamilton cycle.

\[ \begin{align*}
K_3 & \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (y) -- (z) -- (w); \\
\end{tikzpicture}
\end{array} \\
K_4 & \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (y) -- (z) -- (w) -- (x); \\
\end{tikzpicture}
\end{array} \\
K_5 & \quad \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node at (0,0) (x) {x};
  \node at (1,1) (y) {y};
  \node at (2,0) (w) {w};
  \node at (1,2) (z) {z};
  \draw (x) -- (y) -- (z) -- (w) -- (x); \\
\end{tikzpicture}
\end{array}
\end{align*} \]

Theorem 5.8. A complete graph \( K_n \) has

\[ \frac{(n - 1)!}{2} \]

different Hamilton cycles.

Example 5.14. In \( K_4 \) we have \( \frac{(4-1)!}{2} = 3 \) different Hamilton cycle.
Example 5.15. • In $K_5$ we have $\frac{(5-1)!}{2} = 12$ different Hamilton cycles.

• In $K_{20}$ we have $\frac{19!}{2} > 10^{17}$ different Hamilton cycles.

Travelling salesman problem

The salesman has to visit several cities (each exactly once) and returns to the city, which he set off. Moreover his journey should be the shortest, or cheapest, or fastest. There are distance (cost or time) of routing between each pair of cities. For salesmen should be designated such a route, that he can visit each city exactly once and his total distance (cost or time) of the journey was as short as possible (lowest). The problem can be express in complete graph $K_n$ and then find the shortest (cheapest and fastest) Hamiltonian cycle.

Example 5.16. Consider a graph

\begin{verbatim}
   cycle $a, b, c, d, e, a$ weight 230
   cykl $a, b, c, d, a$ weight 110
\end{verbatim}

Theoretically, you can solve the traveling salesman problem by setting $\frac{(n-1)!}{2}$ Hamiltonian cycles and choosing the smallest total weight. Therefore we can use

• the direct method that generates an the exact solution, but only for small $n$. 

Mathematical basis of logistics
• the approximate method, which generates solution close to the optimal, but working quickly.

5.2 Graph Representation

5.2.1 Matrix representation

Definition 5.20. Consider an undirected graph $G = (V, E)$ and $|V| = n$. Let us define a square matrix $A(G) = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 
\text{the number of edges connecting the vertex } x_i \text{ with the vertex } x_j \\
0 \text{ if there is no an edge from } x_i \text{ to } x_j 
\end{cases}$$

Such matrix we called an adjacency matrix for the graph.

Remark 5.7. All definitions in this section are also used for multigraphs.

Example 5.17. Consider the graph
Chapter 5. Graph theory

The adjacency matrix for this graph is of the form

\[
A(G) = \begin{bmatrix}
    x_1 & x_2 & x_3 & x_4 & x_5 \\
    x_1 & 0 & 2 & 0 & 1 & 3 \\
x_2 & 2 & 1 & 0 & 1 & 0 \\
x_3 & 0 & 0 & 0 & 1 & 0 \\
x_4 & 1 & 1 & 1 & 1 & 1 \\
x_5 & 3 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

**Definition 5.21.** Consider an digraph \(G = (V, E)\) and \(|V| = n\). Let us define a square matrix \(A(G) = [a_{ij}]_{n \times n}\)

\[
a_{ij} = \begin{cases} 
    \text{the number of arc connecting the vertex } x_i \text{ with the vertex } x_j \\
    0 \text{ if there is no an arc from } x_i \text{ to } x_j
\end{cases}
\]

Such matrix we called an adjacency matrix for the digraph.

**Example 5.18.** Consider the graph

![Diagram](image.png)

The adjacency matrix for this graph is of the form

\[
A(G) = \begin{bmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    x_1 & 0 & 0 & 0 & 1 \\
x_2 & 2 & 1 & 0 & 1 \\
x_3 & 0 & 0 & 0 & 3 \\
x_4 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

**Remark 5.8.** In many applications we use the square, weight matrix \(W(G) = [w_{ij}]_{n \times n}\)

\[
w_{ij} = \begin{cases} 
    \text{the weight of an edge from the vertex } x_i \text{ to the vertex } x_j \\
    0 \text{ if there is no an edge from } x_i \text{ to } x_j
\end{cases}
\]
Theorem 5.9. Let $G$ be a graph (undirected or directed) with the adjacency matrix $A$. Then $a^k_{i,j}$ gives the number of trails of length $k$ in $G$ form the vertex $i$ to the vertex $j$.

5.2.2 Adjacency list

The list representation is list of the list, for each node a sorted list of adjacent nodes is stored.

Definition 5.22. An adjacency list $Adj$ of a graph (directed or undirected) is a list of graph’s vertices $x_i \in V$ and for every vertex we have the list of his neighbours

$$Adj[x_i] = \{u : \{x_i, u\} \in E\}$$

and for digraph

$$Adj[x_i] = \{u : (x_i, u) \in E\}$$

Example 5.19. Consider a graph

```
1 : 1, 2, 2, 4
2 : 1, 1, 3, 4
3 : 2, 4, 4, 5
4 : 1, 2, 3, 3
5 : 3
```

Example 5.20. Consider a graph
The adjacency list for this digraph is of the form

\begin{align*}
x &: y \\
y &: z \\
z &: x, u \\
u &: x, y
\end{align*}

### 5.3 Shortest paths

One of the most common applications of graphs in logistics, computer science is representing graphs for traffic, distribution or data communication. Let $G$ be a graph with weight function $w$.

**Definition 5.23.** The weight of path $p = (v_0, v_1, ..., v_k)$ is the sum of the weight of edges from the path.

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

The path $p$ we can denote by $v_0 \rightsquigarrow v_k$.

**Definition 5.24.** The shortest-path weight from the vertex $u$ to the vertex $v - \delta(u, v)$ is equal

$$\delta(u, v) = \begin{cases} 
\min\{w(p) : u \rightsquigarrow v\} & \text{if there is a path from } u \text{ to } v \\
\infty & v \text{ is not reachable from } u
\end{cases}$$

**Example 5.21.** The small company has a head office and four points of distribution. The cost of connections between them is presented on the graph.
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The shortest path represents the cost which yields the smallest as possible.

So we have

\[ \delta(ho, a) = 20 \quad \delta(ho, b) = 12 \]
\[ \delta(ho, c) = 18 \quad \delta(ho, d) = 11 \]

The above example shortest paths problem is a type of single-source shortest paths. Of course here the vertex of head office is the source. Usually there are two types of shortest paths problem

- **the single-source shortest path** - when we looking for the shortest path from one chosen vertex - the source, to other vertices or to the selected vertex,

- **the all pairs shortest paths** - when we looking for the shortest paths between all vertices at the same time

**Definition 5.25.** Let \( s \) be a chosen vertex in a graph (source). We usually denote

\[ d(u) = \delta(s, u) \]

The following theorem gives the optimal principle.

**Theorem 5.10.** Let \( G(V, E, w) \). Let \( p = (v_0, v_1, ..., v_k) \) be a shortest path from the vertex \( v_0 \) to the vertex \( v_k \). Then for any \( i \) and \( j \), such that \( 0 \leq i \leq j \leq k \), the subpath \( \tilde{p} = (v_i, v_{i+1}, ..., v_j) \) is a shortest path from \( v_i \) to \( v_j \).
By above principle we can characterized the shortest paths with start vertex \( s \).

**Theorem 5.11.** Let \( p \) be a path from \( s \) to \( v \) in graph \( G \). Then \( p \) is a shortest path if and only if the following condition holds

\[
d(v) \leq d(u) + w(\{u, v\}) \text{ or for digraph } d(v) \leq d(u) + w((u, v))
\]

Let us present the following example taken from [Jun].

**Example 5.22.** A trading ship travels from port \( a \) to port \( b \), where the route (and possible intermediary ports) may be chosen freely. The routes are represented by trails in a digraph \( G \), and the length \( w(e) \) of an edge \( e \) signifies the profit gained by going from \( x \) to \( y \). For some edges, the ship might have to travel empty so that \( w(e) \) is negative for these edges: the profit is actually a loss. Replacing \( w \) by \(-w\) in this network, the shortest path represents the route which yields the largest possible profit.

### 5.3.1 Algorithm of Dijkstra

Now we present the most popular algorithm for finding single-source shortest paths the algorithm of Dijkstra. The length of the shortest paths from a chosen vertex \( s \) are calculated and placed in the array \( d \).

\[
\text{DIJKSTRA}(G, w, s, d)
\]

1. \( d(s) \leftarrow 0 \), \( T \leftarrow V \)
2. for \( v \in V \setminus \{s\} \)
3. \hspace{1em} do \( d(v) \leftarrow \infty \)
4. while \( T \neq \emptyset \)
5. \hspace{1em} do
6. \hspace{2em} find some \( u \in T \) such that \( d(u) \) is minimal
7. \hspace{2em} \( T \leftarrow T \setminus \{u\} \)
8. \hspace{1em} for \( v \in Adj[u] \) and \( v \in T \)
9. \hspace{2em} \hspace{1em} do \( d(v) \leftarrow \min\{d(v), d(u) + w_{uv}\} \)

**Example 5.23.** Consider the graph
with source $s = ho$ find the shortest paths to other vertices.

**Solution.** For our graph, the adjacency list is of the form

\[
\begin{align*}
ho &: a, b, d \\
a &: b, c, ho \\
b &: a, c, d, ho \\
c &: a, b, d \\
d &: b, c, ho
\end{align*}
\]

- for steps 1-3 we have, that $T = \{ho, a, b, c, d\}$ and

\[
\begin{array}{c|c|c|c|c|c}
\hline
u \in V & ho & a & b & c & d \\
\hline
d(u) & 0 & \infty & \infty & \infty & \infty \\
\hline
\end{array}
\]

- the first iteration of while loop because $d(ho) = 0$ is minimal and $ho \in T$, so $u = ho$, $T = \{a, b, c, d\}$ and we can calculate $d(a), d(b), d(d)$.

\[
\begin{array}{c|c|c|c|c|c}
\hline
u \in V & ho & a & b & c & d \\
\hline
d(u) & 0 & 24 & 12 & \infty & 11 \\
\hline
\end{array}
\]

- the second iteration of while loop because $d(d) = 11$ is minimal and $d \in T$, so $u = d$, $T = \{a, b, c\}$ and

\[
\begin{array}{c|c|c|c|c|c}
\hline
u \in V & ho & a & b & c & d \\
\hline
d(u) & 0 & 24 & 12 & 18 & 11 \\
\hline
\end{array}
\]

- the third iteration of while loop because $d(b) = 12$ is minimal and $b \in T$, so $u = b$, $T = \{a, c\}$ and

\[
\begin{array}{c|c|c|c|c|c}
\hline
u \in V & ho & a & b & c & d \\
\hline
d(u) & 0 & 20 & 12 & 18 & 11 \\
\hline
\end{array}
\]
the next two iterations do not changes the values in an array \( d \) and \( T = \emptyset \).

We compute the length of the shortest paths in the array \( d \).

### 5.4 Minimum spanning tree

This section is devoted to a very important optimization problem - finding a tree in our model, which contains all vertices and for which the sum of all edge weights is minimal. This problem has many applications for example, the vertices might represent warehouses or stores we want to connect. Then the edges represent the possible connections and the length of an edge states how much it would cost to build that connection. Let \( G(V, E) \) be a connected graph.

**Definition 5.26.** The tree \( T(V_T, E_T) \) we called a spanning tree if \( T \) is subgraph of \( G \), \( T \subset G \) and \( V_G = V_T \).

**Example 5.24.** Graphs on the picture b) are spanning trees for graph \( G \).

**Definition 5.27.** The weight of a tree \( T = (V_T, E_T, w) \) we called

\[
w(T) = \sum_{e \in E_T} w(e)
\]

**Definition 5.28.** The \( T \) is minimal spanning tree of \( G \), if \( T \) is a spanning tree of \( G \) and \( w(T) \rightarrow \text{min} \).

**Example 5.25.** Consider the graph
Then trees on the picture

are two spanning trees of our graph $G$ and $w(T_1) = 12$ and $w(T_2) = 15$. It easy to notice, that $T_1$ is a minimal spanning tree.

### 5.4.1 Algorithm of Kruskal

Now we present a very popular algorithm due to Kruskal. Let $G = (V, E, w)$ be a connected graph with $V = \{1, \ldots, n\}$ and $E = \{e_1, \ldots, e_m\}$. For the purpose of the algorithm, edges should be sorted in ascending order by weight

$$w(e_1) \leq w(e_2) \leq \ldots w(e_m)$$

**Kruskal**($G, w, T$)

1. $T \leftarrow \emptyset$
2. for $k = 1$ to $m$
3. do if the edge $e_k$ does not form a cycle
4. then $T \leftarrow T \cup \{e_k\}$

**Example 5.26.** Consider the graph of the previous example
First of all we must sort the edges

\[
\begin{align*}
    w(\{a, b\}) &\leq w(\{d, c\}) &\leq w(\{f, g\}) &\leq w(\{a, d\}) &\leq w(\{e, f\}) \leq \\
    w(\{d, e\}) &\leq w(\{c, b\}) &\leq w(\{f, c\}) &\leq w(\{g, h\}) &\leq w(\{g, b\})
\end{align*}
\]

So

\[
\begin{align*}
e_1 &= \{a, b\}, & e_2 &= \{d, c\}, & e_3 &= \{f, g\}, & e_4 &= \{a, d\}, & e_5 &= \{e, f\}, \\
e_6 &= \{d, e\}, & e_7 &= \{c, b\}, & e_8 &= \{f, c\}, & e_9 &= \{g, h\}, & e_{10} &= \{e, h\}, & e_{11} &= \{g, b\}
\end{align*}
\]

Therefore, doing successive iterations we obtain

We also can not add the edges \(e_7 = \{b, c\}\) and \(e_{11} = \{g, b\}\). Using the Kruskal algorithm we pointed the spanning tree in the graph.
5.5 Maximum flow problem

Consider a network of pipelines that transport natural gas from wells to industrial customers. A network is shown in Fig.

Each pipe has a finite maximum capacity. How can we determine the maximum network capacity of the network between wells $w_1$, $w_2$ and industrial customers $ic_1$, $ic_2$, $ic_3$?

In the language of graph theory our a network of pipelines network is networks of the following special kind.

**Definition 5.29.** Let $G = (V, E)$ be a weight digraph where weight we called capacity, $c : E \to \mathbb{R}_+ \cup \{0\}$. Moreover, let $s$ and $t$ be two special vertices of $G$ such that $t$ is accessible from $s$. The vertex $s$ we called source and $t$ - sink. Then $N = (G, c, s, t)$ is called a flow network with source $s$ and sink $t$.

By this definition we have if $(u, w) \in E$, then $c(u, w) \geq 0$. But usually we assume more, that also if $(u, w) \notin E$, then $c(u, w) = 0$.

In our previous example we must add two vertices. The source $s$ we have to connect with vertices $w_1$ and $w_2$ and set up a sufficiently large capacity and respectively $ic_1$, $ic_2$, $ic_3$ with $t$. 

**Definition 5.30.** The flow \( f \) on flow network is a mapping \( f : E \rightarrow \mathbb{R}_+ \cup \{0\} \), which satisfying the following conditions

(F1) for all edges \((u, w) \in E\)
\[
0 \leq f(u, w) \leq c(u, w).
\]
This condition is called the capacity condition it says that for any edge flow we can not exceed the capacity of this edge.

(F2) The condition for the source \( s \):
\[
\sum_u f(s, u) - \sum_u f(u, s) = F(f),
\]
where \( F \) is the value of the flow.

(F3) The condition for the sink \( t \):
\[
\sum_u f(t, u) - \sum_u f(u, t) = -F(f).
\]

(F4) For all vertices \( v \in V \setminus \{s, t\} \)
\[
\sum_u f(u, v) - \sum_u f(v, u) = 0.
\]
The last condition is the condition of the flow, which says, that the total flow flowing into the vertex \( u \notin \{s, t\} \) is equal to the total outgoing flow from the vertex.

**Example 5.27.** For instance in the part of the flow network

for the vertex \( b \in V \)
\[
\sum_u f(u, b) = f(a, b) = 5
\]
\[
\sum_u f(b, u) = f(b, c) + f(b, d) = 2 + 3
\]
**Definition 5.31.** The flow $f^*$ is called maximal flow if

$$F(f^*) = \max_{f \in F_N} F(f),$$

where $F_N$ is a set of value of all admissible flows in $N$.

**Example 5.28.** For our network of pipelines, the value of flow is indicated on each edge (flow/capacity).

The maximum flow $F$ is equal 36.

**Definition 5.32.** Let $f$ be a flow in the network $N$. A path $p$ is called an augmenting path with respect to $f$ if $f(u, v) < c(u, v)$ holds for every forward edge $(u, v) \in p$ and $f(u, v) > 0$ for every backward edge $(u, v) \in p$.

While investigating the maximum flow, we use the residual capacity.

**Definition 5.33.** The residual capacity $c_f$ for the edge $(u, v)$ from the augmenting path $p$ is equal

$$c_f(u, w) = \begin{cases} c(u, w) - f(u, w) & \text{if the edge } (u, w) \text{ is forward,} \\ f(u, w) & \text{if the edge } (u, w) \text{ is backward.} \end{cases}$$

For chosen augmenting path $p$ we define

**Definition 5.34.** The residual capacity of an augmenting path $p$ is equal

$$c_f(p) = \min\{c_f(u, w) : (u, w) \in p\}.$$
Knowledge of the augmenting path enables us to increase the value of flow of \( c_f(p) \). So we can create a new network of increased flow of \( f' \), where

\[
f'(u, w) = \begin{cases} 
  f(u, w) + c_f(p) & \text{if the edge } (u, w) \text{ is forward}, \\
  f(u, w) - c_f(p) & \text{if the edge } (u, w) \text{ is backward}, \\
  f(u, w) & \text{if the edges } (u, w), (w, u) \text{ do not belong to } p.
\end{cases}
\]

**Theorem 5.12.** The flow \( f \) on a network \( N = (G, c, s, t) \) is maximal if and only if there are no augmenting paths with respect to \( f \).

### 5.5.1 Algorithm of Ford-Fulkerson

1. find augmenting path \( p \) with respect to \( f \) from source \( s \) to sink \( t \).

2. for every edge \( (u, w) \) from \( p \) compute the residual capacity

\[
c_f(u, w) = \begin{cases} 
  c(u, w) - f(u, w) & \text{if the edge } (u, w) \text{ is forward}, \\
  f(u, w) & \text{if the edge } (u, w) \text{ is backward}.
\end{cases}
\]

3. compute

\[
c_f(p) = \min \{ c_f(u, w) : (u, w) \in p \}.
\]

4. set a new network flow \( f' \)

\[
f'(u, w) = \begin{cases} 
  f(u, w) + c_f(p) & \text{if the edge } (u, w) \text{ is forward}, \\
  f(u, w) - c_f(p) & \text{if the edge } (u, w) \text{ is backward}, \\
  f(u, w) & \text{if the edges } (u, w), (w, u) \text{ do not belong to } p.
\end{cases}
\]

5. compute the current flow \( F \) in the network

\[
F = F + c_f(p).
\]

6. if there are augmenting path \( p \), then go to step 1, otherwise - to 7

7. stop

**Example 5.29.** Determine the maximum flow for the network \( N \)
The capacity in our network are the following

<table>
<thead>
<tr>
<th>edge e</th>
<th>(s, x)</th>
<th>(s, z)</th>
<th>(s, u)</th>
<th>(x, y)</th>
<th>(x, v)</th>
<th>(y, t)</th>
<th>(z, y)</th>
<th>(z, v)</th>
<th>(u, v)</th>
<th>(v, t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>c(e)</td>
<td>7</td>
<td>10</td>
<td>9</td>
<td>3</td>
<td>5</td>
<td>12</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

In the table below there are collected flows for every edge $f_i(e)$, $e \in E$ for $i$-th iteration. In the last row we have the final flow for each $e \in E$ and the maximum flow $F$ in the network $N$. At the beginning we assume that $f(e) = 0$, $e \in E$, which is in the first row and $F = 0$. Let $e$ be any edge

<table>
<thead>
<tr>
<th></th>
<th>(s, x)</th>
<th>(s, z)</th>
<th>(s, u)</th>
<th>(x, y)</th>
<th>(x, v)</th>
<th>(y, t)</th>
<th>(z, y)</th>
<th>(z, v)</th>
<th>(u, v)</th>
<th>(v, t)</th>
<th>$e \in p_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(e)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_1(e)$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$f_2(e)$</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$f_3(e)$</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$f_4(e)$</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>$f_5(e)$</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$f(e)$</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>14</td>
<td>$F = 21$</td>
</tr>
</tbody>
</table>

**Iteration I.**

1. Let $p$ be an augmenting path from $s$ to $t$, $p = \{(s, x), (x, y), (y, t)\}$.

2.
3. \( c_f(p) = \min(7, 3, 12) = 3 \).

4.

<table>
<thead>
<tr>
<th></th>
<th>( (s, x) )</th>
<th>( (x, y) )</th>
<th>( (y, t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(e) )</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

5. \( F = F + c_f(e) = 3 \).

### Iteration II.

1. Let \( p = \{ (s, z), (z, y), (y, t) \} \).

2.

<table>
<thead>
<tr>
<th></th>
<th>( (s, z) )</th>
<th>( (z, y) )</th>
<th>( (y, t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_f(e) )</td>
<td>10</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

3. \( c_f(p) = \min(10, 4, 9) = 4 \).

4.

<table>
<thead>
<tr>
<th></th>
<th>( (s, z) )</th>
<th>( (z, y) )</th>
<th>( (y, t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(e) )</td>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

5. \( F = F + c_f(e) = 3 + 4 = 7 \).
Chapter 5. Graph theory

Iteration III.

1. Let $p = \{(s, x), (x, v), (v, t)\}$.

2.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$(s, x)$</th>
<th>$(x, v)$</th>
<th>$(v, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_f(e)$</td>
<td>4</td>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

3. $c_f(p) = \min(4, 5, 14) = 4$.

4.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$(s, x)$</th>
<th>$(x, v)$</th>
<th>$(v, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(e)$</td>
<td>7</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

5. $F = F + c_f(e) = 7 + 4 = 11$.

Iteration IV.

1. Niech $p = \{(s, u), (u, v), (v, t)\}$.

2.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$(s, u)$</th>
<th>$(u, v)$</th>
<th>$(v, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_f(e)$</td>
<td>9</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

3. $c_f(p) = \min(9, 5, 10) = 5$.

4.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$(s, u)$</th>
<th>$(u, v)$</th>
<th>$(v, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(e)$</td>
<td>5</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

5. $F = F + c_f(e) = 11 + 5 = 16$. 
Iteration V.

1. And finally \( p = \{(s, z), (z, v), (v, t)\} \).

2.

\[
\begin{array}{c|ccc}
 e & (s, z) & (z, v) & (v, t) \\
 \text{c}_f(e) & 6 & 6 & 5 \\
\end{array}
\]

3. \( c_f(p) = \min(6, 6, 5) = 5 \).

4.

\[
\begin{array}{c|ccc}
 e & (s, z) & (z, v) & (v, t) \\
 \text{f}'(e) & 9 & 5 & 14 \\
\end{array}
\]

5. \( F = F + c_f(e) = 16 + 5 = 21 \).

There no augmenting path in the network, so we obtain the flows on every edge. The maximum flow is \( F = 21 \).
5.6 Review Exercises

**Exercise 5.1.** Draw a graph $G(V, E)$, where $V = \{a, b, c, d, e, f, g, h\}$ and

$$E = \{(a, b), (a, e), (a, c), (b, c), (c, f), (g, h), (f, g), (d, c), (b, d), (f, d), (f, h)\}$$

Give the adjacency list $Adj$ of this graph.

**Exercise 5.2.** Draw a digraph $G(V, E)$, where $V = \{a, b, c, d, e\}$ and

$$E = \{(a, b), (b, c), (c, d), (d, e), (e, c), (b, e), (a, c)\}$$

Find $\Delta$ and write $deg_{in}(v)$ for all $v \in V$. Does exist a direct trail from $a$ to $b$?

**Exercise 5.3.** If it is possible, draw a graph of the given sequence of vertex $(2, 2, 6, 2, 2)$.

**Exercise 5.4.** The graph $G$ has 26 edges, 9 vertices of the degree 1, 3 vertices of degree 2, 8 vertices of degree 4 and the other vertices are the degree of 5. How many vertices are in the graph $G$?

**Exercise 5.5.** Let

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 
\end{bmatrix}
$$

be an adjacency matrix of digraph. Draw a digraph.

**Exercise 5.6.** How many vertices has a complete graph, which possesses 28 edges?

**Exercise 5.7.** Consider graphs
In which graph there is an Eulerian cycle or trail?

Exercise 5.8. Show, for every tree with \( n \) vertices, the sum of degree of vertices is equal \( 2n - 2 \).

Exercise 5.9. In some tree we have 2 vertices of degree 4, one of 3, one of degree 2 and the other vertices are the degree of 1. How many vertices are in the tree?

Exercise 5.10. The table shows the distances between cities in the (km). The company Orlen based in the Płock wants to connect the pipeline.

<table>
<thead>
<tr>
<th></th>
<th>Łódź</th>
<th>Sieradz</th>
<th>Skierniewice</th>
<th>Płock</th>
<th>Piotrków T.</th>
<th>Konin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Warszawa</td>
<td>137</td>
<td>210</td>
<td>76</td>
<td>110</td>
<td>135</td>
<td>227</td>
</tr>
<tr>
<td>Łódź</td>
<td>61</td>
<td>62</td>
<td>103</td>
<td>48</td>
<td>111</td>
<td></td>
</tr>
<tr>
<td>Sieradz</td>
<td></td>
<td></td>
<td>144</td>
<td>77</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td>Skierniewice</td>
<td></td>
<td></td>
<td>91</td>
<td>84</td>
<td>166</td>
<td></td>
</tr>
<tr>
<td>Płock</td>
<td></td>
<td></td>
<td>163</td>
<td></td>
<td></td>
<td>123</td>
</tr>
<tr>
<td>Piotrków Tryb.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>173</td>
<td></td>
</tr>
</tbody>
</table>

How many km of pipes needed to carry out this venture?

Exercise 5.11. The central warehouse in retail chain is at the point \( A \). Determine the shortest connection to the other warehouses. The table of distances

\[
\begin{array}{c|cccccccc}
A & B & C & D & E & F & G \\
\hline
A & - & 1 & 2 & \infty & \infty & \infty & \infty \\
B & & - & \infty & 4 & 3 & \infty & \infty \\
C & & & - & 1 & 5 & 7 & \infty \\
D & & & & - & 3 & 8 & \infty \\
E & & & & & - & 1 & 3 \\
F & & & & & & - & 3 \\
G & & & & & & & - \\
\end{array}
\]

Exercise 5.12. Find the maximum flow in the network.
Chapter 5. Graph theory
Chapter 5. Graph theory

Answeres

Exercise 5.1. The graph’s picture:

The adjacency list $\text{Adj}$:

- $a : b, c, e,$
- $b : a, e, c, d$
- $c : a, b, d, f$
- $d : b, c, f,$
- $e : a, b,$
- $f : c, d, g, h$
- $g : f, h,$
- $h : f, g.$

Exercise 5.2. The digraph’s picture:

$\Delta = 4$, $\text{degin}(a) = 0$, $\text{degin}(b) = 1$, $\text{degin}(c) = 2$, $\text{degin}(d) = 1$, $\text{degin}(e) = 3$. Yes.

Exercise 5.3.
Exercise 5.4. \(|E_G| = 26\) we are looking for \(|V|=?\) Let \(x\) be a number of vertices of degree 5. Because of the property \(\sum_{v \in V} \text{deg}(v) = 2|E|\), we have

\[
2|E| = \sum_{v \in V} \text{deg}(v)
\]

\[
2 \cdot 26 = 1 \cdot 9 + 2 \cdot 3 + 4 \cdot 8 + 5 \cdot x
\]

\[
52 = 9 + 6 + 32 + 5x
\]

\[
52 = 47 + 5x
\]

\[
5x = 5
\]

\[
x = 1
\]

therefore \(|V| = 9 + 3 + 8 + 1 = 21\).

Exercise 5.5.

Exercise 5.6. The complete graph \(K_n\) has \(|E| = \frac{n(n-1)}{2}\) edges. Therefore

\[
\frac{n(n-1)}{2} = 28
\]

\[
n^2 - n - 56 = 0
\]

\[
n = 8
\]

So we have \(|V_{K_8}| = 8\).

Exercise 5.7. The first graph has an Eulerian trail because there are two vertices \(d, c\) of odd degree. The Eulerian trail must start in \(d\) or \(c:dbadcaecfedfbc\). The second has an Eulerian cycle: \(dbaceadefcb\).

Exercise 5.8. Let be a tree \(T(V_T, E_T)\), the we have \(|V_T| = n\) and \(|E_T| = n - 1\). Because of \(\sum_{v \in V} \text{deg}(v) = 2|E|\), we have \(\sum_{i=1}^{n} \text{deg}(v) = 2(n - 1) = 2n - 2\).
Exercise 5.9. Let \( x \) be a number of vertices of degree one. In this tree \( T \) we have \( V_T = x + 1 + 1 + 2 = x + 4 \) vertices. Because of previous exercise, we have

\[
4 \cdot 2 + 3 \cdot 1 + 2 \cdot 1 + 1 \cdot x = 2(x + 4) - 2
\]

\[
x = 7
\]

Therefore we have \( V_T = x + 4 = 11 \).

Exercise 5.10. 459

Exercise 5.11. Graph for our retail chain is of the form

![Graph Diagram]

The shortest connections are

<table>
<thead>
<tr>
<th>( u \in V )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d(u) )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

The shortest paths in our network is the following

![Graph Diagram with Shortest Paths]

Exercise 5.12. Maximum flow \( F = 20 \).
Chapter 5. Graph theory
Chapter 6

Test yourself

6.1 Applied calculus

1. For the function \( f(x) = -2x + 5 \) if \( 1 \leq x \leq 4 \)

   a) the domain is \( D_f = [1, 4] \),
   b) the domain is \( D_f = [-3, 3] \),
   c) the range is \([-3, 3]\).

   a) \[ \text{YES} \quad \text{NO} \]  
   b) \[ \text{YES} \quad \text{NO} \]  
   c) \[ \text{YES} \quad \text{NO} \]

2. Which of the domain is determined correctly

   a) \( f(x) = \sqrt{x + 1} \), the domain is \( D_f = [1, \infty) \),
   b) \( f(x) = x^2 + 3x \), the domain is \( D_f = \mathbb{R} \),
   c) \( f(x) = \frac{3x^2 + 2}{x^2 - 4x + 6} \), the domain is \( D_f = \mathbb{R} \setminus \{1, 3\} \).

   a) \[ \text{YES} \quad \text{NO} \]  
   b) \[ \text{YES} \quad \text{NO} \]  
   c) \[ \text{YES} \quad \text{NO} \]

3. The difference quotient of the function \( f(x) = x^2 \) is of the form

   a) \( \frac{(x^2+\Delta x)^2-x^2}{\Delta x} \),
   b) \( 2x + \Delta x \),
   c) \( \frac{(x^2-\Delta x)^2-\Delta x}{\Delta x} \).

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Chapter 6. Test yourself

4. Which of the limit is calculated correctly

a) \( \lim_{x \to 4} \frac{x - 4}{x^2 - 16} = \frac{1}{8} \),

b) \( \lim_{x \to 0} \frac{x^3 + 2x^2 + 4x}{3x^2 + 2x} = 0 \),

c) \( \lim_{x \to -5} \frac{x^2 - 5x}{x + 2} = 0 \).

5. Which of the limit is calculated correctly

a) \( \lim_{x \to \infty} (1 + \frac{2}{x})^x = e^2 \),

b) \( \lim_{x \to 0} \frac{\sin 7x}{x} = 7 \),

c) \( \lim_{x \to \infty} \frac{2x^3 + 4x^2 - 5x}{x^3 - 1} = 2 \).

6. The function

\[
f(x) = \begin{cases} 
  x^2 + x + 1 & \text{for } x \geq 1 \\
  -2x + 5 & \text{for } 1 < x 
\end{cases}
\]

a) is continuous for \( x = 1 \),

b) is continuous for all \( x \) from the domain,

c) hasn’t a limit for \( x = 1 \).

7. Which of the derivative is calculated correctly

a) \( (3x^{18} - 5x + 6x^{-3})' = 54x^{17} - 5 + 18x^{-4} \),

b) \( (5e^x + 2 \sin x)' = 5e^x + 2 \cos x \),

c) \( (-2x^2 - \ln x + \sqrt{5})' = -4x - x^{-1} \).
8. Which of the derivative is calculated correctly

\[ \frac{x+1}{x+4} = \frac{3}{(x+4)^2}, \]
\[ (x^3 \cos x)' = -3x^2 \sin x, \]
\[ (xe^x)' = e^x(x+1). \]

a) YES  NO  b) YES  NO  c) YES  NO

9. The function \( f(x) = x^3 - 3x + 3 \)

a) is increasing for \( x \in (1, \infty) \),

b) is decreasing for \( x \in (-\infty, -1) \),

c) has a local minimum for \( x = -1 \).

a) YES  NO  b) YES  NO  c) YES  NO

10. For the function \( f(x) = x^3 - 3x^2 + 1 \) and \( x \in [0, 1] \) has

a) the smallest value is \(-3\),

b) the smallest value is \(-1\),

c) the largest value is \(1\).

a) YES  NO  b) YES  NO  c) YES  NO

11. The function \( f(x) = x^4 - 24x^2 + 8 \)

a) is convex for \( x \in (-2, 2) \),

b) is concave for \( x \in (2, \infty) \),

c) the inflection points are \(-2, 2\).

a) YES  NO  b) YES  NO  c) YES  NO

12. Which of the integral is calculated correctly

\[ \int (9x^2 - 7)dx = 3x^3 - 7x + C, \]
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b) \[ \int \frac{9x^9 + 5x}{x} \, dx = x^8 + 5x^2 + C, \]

c) \[ \int x^3 e^{(x^4 - 10)} \, dx = \frac{1}{4} e^{(x^4 - 10)} + C. \]

a) YES  NO  b) YES  NO  c) YES  NO

13. Which of the definite integral is calculated correctly

a) \[ \int_{1}^{2} (4x + 3) \, dx = 19, \]

b) \[ \int_{0}^{2} x \, dx = \frac{1}{2} x^2, \]

c) \[ \int_{1}^{e} \frac{x^2 + x + 1}{x} \, dx = \frac{1}{2} e^2 + e - \frac{1}{2}. \]

a) YES  NO  b) YES  NO  c) YES  NO

14. The field bounded by \( f(x) = 2 - x^2 \) and \( g(x) = x^2 \)

a) 0,

b) \( \frac{8}{3} \),

c) 2.

a) YES  NO  b) YES  NO  c) YES  NO
Answers

1. YES, NO, YES; 2. NO, YES, YES; 3. YES, YES, NO; 4. YES, NO, YES; 5. YES, YES, YES; 6. YES, YES, NO;
7. NO, YES, YES; 8. YES, NO, YES; 9. YES, NO, NO; 10. YES, NO, YES; 11. NO, NO, YES; 12. YES, NO, YES;
13. NO, NO, YES; 14. NO, YES, NO;
6.2 Algebra

1. For the matrices $A = \begin{bmatrix} 0 & 10 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

   a) $A + B = \begin{bmatrix} -2 & 10 \\ 2 & 2 \end{bmatrix}$,
   
   b) $A - 5B = \begin{bmatrix} 10 & 10 \\ 2 & -10 \end{bmatrix}$,
   
   c) $B = 2I$.

   a) YES NO    b) YES NO    c) YES NO

2. For the matrices $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

   a) $A \cdot B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & 2 & 8 \end{bmatrix}$,
   
   b) the multiplication $B \cdot A$ is not feasible,
   
   c) $A^T \cdot B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & 2 & 8 \end{bmatrix}$.

   a) YES NO    b) YES NO    c) YES NO

3. For the matrix $A = \begin{bmatrix} 4 & 5 & 0 \\ 1 & -5 & -2 \\ 0 & 0 & 10 \end{bmatrix}$

   a) the sum of elements on the main diagonal is equal 9,
   
   b) the sum of elements on the second column is equal 0,
   
   c) $|A| = -250$.

   a) YES NO    b) YES NO    c) YES NO
4. For the matrix \( A = [a_{i,j}] = \begin{bmatrix} 4 & 5 & 0 & -1 \\ 1 & -5 & -2 & -4 \\ 3 & 4 & 10 & 2 \\ 0 & -7 & 0 & 2 \end{bmatrix} \)

a) \( \sum_{k=1}^{4} a_{3k} = 19 \),

b) \( \sum_{k=1}^{4} a_{k3} = 8 \),

c) \( \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} = 12 \).

5. For the matrix \( A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \)

a) \( \det A = 10 \),

b) \( \det A = 2 \),

c) \( \det A = -2 \).

6. The matrix \( A = \begin{bmatrix} 0 & 3 & 2 & -1 \\ 1 & 0 & 12 & -7 \\ 3 & -4 & 0 & 6 \\ 1 & 1 & 11 & 0 \end{bmatrix} \)

a) is a matrix of order 5,

b) is a diagonal matrix,

c) is a identity matrix.

7. Which of the determinant is compute correctly
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a) \[
\begin{vmatrix}
-2 & 1 \\
1 & -2
\end{vmatrix} = 3,
\]

b) \[
\begin{vmatrix}
2 & 3 & 1 \\
-5 & 0 & 1 \\
2 & 2 & 1
\end{vmatrix} = -22,
\]

c) \[
\begin{vmatrix}
1 & -1 & 0 \\
2 & 3 & -1 \\
1 & -2 & 1
\end{vmatrix} = 4.
\]

a) YES NO b) YES NO c) YES NO

8. The solution for the system is

\[
\begin{cases}
4x_1 + 2x_2 + x_3 + x_4 = 15 \\
x_1 + x_2 - x_3 + x_4 = 4 \\
x_1 + 7x_2 + 2x_3 - 3x_4 = 9 \\
5x_1 - 6x_2 - 3x_3 + x_4 = 18
\end{cases}
\]

a) \(x_1 = 1, x_2 = 2, x_3 = 3, x_4 = -4,\)

b) \(x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4,\)

c) \(x_1 = -1, x_2 = 2, x_3 = 3, x_4 = -4.\)

a) YES NO b) YES NO c) YES NO
Answers

1. YES, NO, NO; 2. YES, YES, NO; 3. YES, YES, YES; 4. NO, NO, YES; 5. NO, NO, YES; 6. YES, NO, NO;
7. YES, NO, YES; 8. NO, YES, NO;
6.3 Graph theory

1. Each of digraph possesses
   a) the vertex, which is the source,
   b) at least one vertex,
   c) the vertex, which is the sink.
   a) YES  NO  b) YES  NO  c) YES  NO

2. The adjacency list of the graph is of the form

   a :  b,  c,  e,
   b :  a,  d,
   c :  a,  d,  e,  f
   d :  b,  c,
   e :  a,  c,  f,
   f :  c,  e,  g,  h
   g :  f,  h,
   h :  f,  g.

   The graph is
   a) a tree,
   b) a complete graph,
   c) a connected graph.
   a) YES  NO  b) YES  NO  c) YES  NO

3. In each graph
   a) the sum of degrees of vertices is equal to double the number of edges,
   b) the number of vertices of even degree is always odd,
   c) we have an edge.
   a) YES  NO  b) YES  NO  c) YES  NO
4. Every complete graph $K_n$
   
   a) has $\frac{n(n-1)}{2}$ edges,
   
   b) is acyclic ,
   
   c) has the same degree for each vertex.

   a) YES  NO  
   b) YES  NO  
   c) YES  NO

5. In the graph the walk $abe$

   a) is a trail,
   
   b) is a path,
   
   c) is a cycle.

   a) YES  NO  
   b) YES  NO  
   c) YES  NO

6. Every tree

   a) is an acyclic graph,
   
   b) is a connected graph,
   
   c) is a complete graph

   a) YES  NO  
   b) YES  NO  
   c) YES  NO

7. In the digraph

   a) YES  NO  
   b) YES  NO  
   c) YES  NO
Chapter 6. Test yourself

a) the vertex $a$ is a source,
b) the vertex $e$ is a sink,
c) $abe$ is a directed path.

a) YES NO b) YES NO c) YES NO

8. In the graph

a) $\Delta = 4$,
b) the sequence of degree is of the form $(0,0,1,4)$,
c) there is an isolated vertex.

a) YES NO b) YES NO c) YES NO

9. Dijkstra’s algorithm helps to solve

a) the traveling salesman problem,
b) the chinese postman problem,
c) the seven bridges of Königsberg problem.

a) YES NO b) YES NO c) YES NO

10. In the network

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a) for the direct edge \((v, y)\) we have the flow 5, the capacity 8,

b) for the direct edge \((x, y)\) we have the flow 3, the capacity 2,

c) there is an augmenting path from \(s\) to \(t\).

a) \[
\begin{array}{cc}
\text{YES} & \text{NO} \\
\end{array}
\]

b) \[
\begin{array}{cc}
\text{YES} & \text{NO} \\
\end{array}
\]

c) \[
\begin{array}{cc}
\text{YES} & \text{NO} \\
\end{array}
\]
Chapter 6. Test yourself

Answers

1. NO, YES, NO; 2. NO, NO, YES; 3. YES, NO, NO; 4. YES, NO, YES; 5. YES, YES, NO; 6. YES, YES, NO;
7. YES, YES, YES; 8. NO, YES, NO; 9. NO, NO, NO; 10. YES, NO, NO;
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