On characterizations of Meir–Keeler contractive maps

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Let \((X,d)\) be a complete metric space and \(T : X \to X\) a map. Suppose there exists a function \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying \(\phi(0) = 0, \ \phi(s) < s\) for \(s > 0\) and that \(\phi\) is right upper semicontinuous such that

\[ d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X. \]

Boyd–Wong [1] showed that \(T\) has a unique fixed point.

Later, Meir–Keeler [2] extended Boyd–Wong’s result to mappings satisfying the following more general condition:

\[ \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \] (1)

In this paper, we characterize condition (1) in terms of a \(\phi\) function as in Boyd–Wong’s theorem. This is obviously desirable since then one can easily see how much more general is Meir–Keeler’s result than Boyd–Wong’s. A characterization was given earlier by Wong [4], but it was in terms of a function \(\delta\) imposed on \(d(Tx, Ty)\) rather than \(d(x, y)\).

**Definition 1.** Let \(\zeta\) be a nondecreasing function from \([0, \infty)\) to \([0, \infty]\). The *pseudo-inverse* of \(\zeta\) is the function defined by

\[ \psi(t) = \sup\{s: \zeta(s) \leq t\} \]

for \(t \in [0, \infty)\).
Proposition 1. Let \( \zeta \) be a nondecreasing function from \([0, \infty)\) to \([0, \infty)\) and \( \psi \) its pseudo-inverse. Then,

1. \( \psi(t) \in [0, \infty) \ \forall t \in [0, \infty) \).
2. \( \psi \) is nondecreasing and right continuous.
3. At every point \( s \) where \( \zeta \) is discontinuous, \( \psi(t) = s \ \forall t \in [\zeta(s^-), \zeta(s^+)) \).

Proof. (1) is obvious. Note that \( \psi(t) = \infty \) if and only if \( \zeta(s) \leq t \) for all \( s \in [0, \infty) \). That \( \psi \) in (2) is nondecreasing is obvious. To prove the right continuity, let \( t_1 > t_2 > \cdots > t_n > \cdots \) and \( \lim_n t_n = t \). Suppose that for large \( n \) \( \psi(t_n) = c \) for some constant \( c \). By the monotonicity of \( \zeta \), we have \( \zeta(s) \leq t_n \ \forall s < c \) for large \( n \); thus, \( \zeta(s) \leq t \ \forall s < c \). This implies that \( \psi(t) \geq c \). But \( \psi(t) \leq \psi(t_n) = c \). So \( \psi(t) = c \) and \( \psi \) is right continuous at \( t \) in this case. So we may assume that all \( \psi(t_n) \) are distinct and hence strictly decreasing. For each \( n \) choose \( s_n \) such that \( \zeta(s_n) = t_n \) and \( \psi(t_{n+1}) < s_n \leq \psi(t_n) \). Then \( s_n \to s = \psi(t^+) \) and \( \zeta(s) \leq \lim_n \zeta(s_n) \leq \lim_n t_n = t \). By the definition of \( \psi(t) \), we then have \( \psi(t) \geq s = \psi(t^+) \).

But \( \psi(t) \leq \psi(t^+) \) by monotonicity. Hence \( \psi(t) = \psi(t^+) \), proving the right continuity.

Let \( r \in (\zeta(s^-), \zeta(s^+)) \). For any \( t > s \), one has \( \zeta(t) \geq \zeta(s^+) > r \) and hence \( \psi(r) \leq t \). Since \( t > s \) is arbitrary, we get \( \psi(r) \leq s \). On the other hand, for any \( t < s \), one has \( \zeta(t) \leq \zeta(s-) \leq r \) which implies \( \psi(r) \geq t \); and since \( t < s \) is arbitrary, \( \psi(r) \geq s \). This proves that \( \psi(r) = s \ \forall r \in (\zeta(s^-), \zeta(s^+)) \). □

Proposition 2. Let \( \zeta \) be a nondecreasing function from \([0, \infty)\) to \([0, \infty)\) and let \( \psi \) be its pseudo-inverse. Let \( X \) be a metric space and \( T : X \to X \). If \( \zeta(d(Tx, Ty)) \leq d(x, y) \ \forall x, y \in X \), then \( d(Tx, Ty) \leq \psi(d(x, y)) \ \forall x, y \in X \); the converse is false.

Proof. If \( \zeta(d(Tx, Ty)) \leq d(x, y) \), then by the definition of pseudo-inverse \( \psi(d(x, y)) \geq d(Tx, Ty) \).

To show that the converse is false, let

\[
\zeta(s) = \begin{cases} 
2s & \text{for } 0 \leq s \leq 1/2, \\
1 & \text{for } 1/2 < s < 1, \\
\infty & \text{for } 1 \leq s < \infty.
\end{cases}
\]

Then,

\[
\psi(t) = \begin{cases} 
t/2 & \text{for } 0 \leq t < 1, \\
1 & \text{for } 1 \leq t < \infty.
\end{cases}
\]

If \( X \) is any discrete metric space in which distance between any two distinct points is 1, and \( T \) the identity map, then \( T \) satisfies the inequality \( d(Tx, Ty) \leq \psi(d(x, y)) \), but not the inequality \( \zeta(d(Tx, Ty)) \leq d(x, y) \). □

Let \( (X, d) \) be a metric space and \( T : X \to X \). The modulus of uniform continuity \( \delta(\varepsilon) \) of \( T \) is defined to be

\[
\delta(\varepsilon) = \sup \{ \lambda : d(x, y) < \lambda \Rightarrow d(Tx, Ty) < \varepsilon \}
\]

for \( \varepsilon > 0 \) and \( \delta(0) = 0 \).
Proposition 3.  (1) \( \delta(\varepsilon) = \inf \{d(x, y): d(Tx, Ty) \geq \varepsilon \} \).
(2) \( \delta(\varepsilon) \) is nondecreasing and \( 0 \leq \delta(\varepsilon) \leq \infty \).
(3) \( \delta(d(Tx, Ty)) \leq d(x, y) \forall x, y \in X \).

Proof. Let \( \mu(\varepsilon) = \inf \{d(x, y): d(Tx, Ty) \geq \varepsilon \} \). Clearly \( \mu(0) = 0 = \delta(0) \). Let \( \varepsilon > 0 \). Call the set in the definition of \( \delta(\varepsilon) \) \( S \). If \( d(x_0, y_0) < \mu(\varepsilon) \), then \( d(x_0, y_0) \notin \{d(x, y): d(Tx, Ty) \geq \varepsilon \} \). So \( d(Tx_0, Ty_0) < \varepsilon \) and \( \mu(\varepsilon) \in S \). If \( \lambda \in S \), then \( d(Tx, Ty) \geq \varepsilon \Rightarrow d(x, y) \geq \lambda \). Thus every number in the definition of \( \mu(\varepsilon) \) is \( \geq \lambda \) and upon taking infimum \( \mu(\varepsilon) \geq \lambda \). This means \( \mu(\varepsilon) = \max S = \delta(\varepsilon) \). That \( \delta(\varepsilon) \) is nondecreasing is obvious. Note that \( \inf \emptyset = \infty \). Incidentally, we proved that for \( \varepsilon > 0 \), the supremum in the definition of \( \delta \) is actually the maximum.

If \( d(Tx, Ty) = 0 \), then (3) is obvious; so assume \( d(Tx, Ty) = \varepsilon > 0 \). If \( d(x, y) < \delta(\varepsilon) \), then \( d(Tx, Ty) < \varepsilon \), a contradiction. So one must have \( d(x, y) \geq \delta(\varepsilon) = \delta(d(Tx, Ty)) \). \( \square \)

Recall that \( T \) is uniformly continuous if
\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } d(x, y) < \delta \Rightarrow d(Tx, Ty) < \varepsilon.
\]

Proposition 4. Let \( X \) be a metric space and \( T: X \rightarrow X \). Let \( \delta(\varepsilon) \) be the modulus of uniform continuity of \( T \). The following are equivalent:

1. \( T \) is uniformly continuous.
2. \( \delta(\varepsilon) > 0 \) for every \( \varepsilon > 0 \).
3. There exists a nondecreasing function \( \zeta: [0, \infty) \rightarrow [0, \infty] \) such that \( \zeta(0) = 0, \zeta(\varepsilon) > 0 \) for every \( \varepsilon > 0 \) and \( \zeta(d(Tx, Ty)) \leq d(x, y) \) for every \( x, y \in X \).
4. There exists a function \( \phi: [0, \infty) \rightarrow [0, \infty] \) such that \( \phi(0) = 0, \phi \) is continuous at \( 0 \) and \( d(Tx, Ty) \leq \phi(d(x, y)) \forall x, y \in X \).

In (3), one can choose \( \zeta \) to be the modulus of uniform continuity of \( T \). In (4), one can choose \( \phi \) to be also nondecreasing and right continuous.

Proof. (1) \( \Rightarrow \) (2) follows from the definition. (2) \( \Rightarrow \) (3): Take \( \zeta = \delta \) and apply Proposition 3. (3) \( \Rightarrow \) (4): Take \( \phi \) to be the pseudo-inverse of \( \zeta \) and apply Proposition 2. Since \( \zeta(0) = 0 \) and \( \zeta(\varepsilon) > 0 \) for \( \varepsilon > 0 \), one has \( \phi(0) = 0 \) by the definition of pseudo-inverse. \( \phi \) is continuous at 0 by item 2 in Proposition 1.

(4) \( \Rightarrow \) (1): \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( r < \delta \Rightarrow |\phi(r) - \phi(0)| = \phi(r) < \varepsilon \). Therefore, \( d(x, y) < \delta \Rightarrow d(Tx, Ty) \leq \phi(d(x, y)) < \varepsilon \). \( \square \)

Definition 2. A function \( \lambda: [0, \infty) \rightarrow [0, \infty) \) will be called an \( L \)-function if \( \lambda(0) = 0, \lambda(s) > 0 \forall s > 0 \), and for every \( s > 0 \), there exists \( u > s \) such that
\[
\lambda(t) \leq s \text{ for } t \in [s, u].
\]

Note that every \( L \)-function satisfies \( \lambda(s) \leq s \forall s > 0 \).
Proposition 5. Let \( \zeta : [0, \infty) \to [0, \infty] \) be nondecreasing, \( \zeta(0) = 0 \), \( \zeta(s) > s \) for \( s > 0 \). Let \( \phi \) be its pseudo-inverse. Then,

1. \( \phi(t) \leq t \quad \forall t \in [0, \infty) \).
2. \( \phi \) is an L-function except that \( \phi(s) \) could be 0 for some \( s > 0 \).

Proof. (1) We have \( \phi(0) = 0. \) If \( t > 0 \) and \( \zeta(s) \leq t \), then \( s < t \) (for if \( s \geq t \), then \( \zeta(s) \geq \zeta(t) > t \)). So \( \phi(t) = \sup \{ s : \zeta(s) \leq t \} \leq t. \)

(2) Let \( t > 0. \) Suppose \( \phi(t) < t. \) Then by the right continuity of \( \phi \) there exists \( u > t \) such that \( \phi(s) < t \quad \forall s \in [t, u]. \)

If \( \phi(t) = t, \) then \( \sup \{ s : \zeta(s) \leq t \} = t. \) This implies that \( \zeta(t-) = t. \) But \( \zeta(t) > t \) by assumption. So \( \zeta \) has a discontinuity at \( t. \) By item 3 in Proposition 1, there exists \( u > t \) such that \( \phi(s) = t \quad \forall s \in [t, u]. \)

If \( \zeta \) is discontinuous at 0, then \( \phi(s) = 0 \quad \forall s \in [0, \zeta(0+)]. \) \( \square \)

The proof of (3) \( \Rightarrow \) (1) in the following theorem is due to Wong [4]; we present it here for completeness.

Theorem 1. Let \( X \) be a metric space. Let \( T : X \to X \) and let \( \delta(\varepsilon) \) be its modulus of uniform continuity. The following are equivalent:

1. \( T \) satisfies the Meir–Keeler’s condition (1).
2. \( \delta(\varepsilon) > \varepsilon \quad \forall \varepsilon > 0. \)
3. There exists a right lower semicontinuous function \( \zeta : [0, \infty) \to [0, \infty) \) such that \( \zeta(0) = 0, \zeta(\varepsilon) > \varepsilon \) for every \( \varepsilon > 0 \) and \( \zeta(d(Tx, Ty)) \leq d(x, y) \) for every \( x, y \in X \) [4].
4. There exists an L-function \( \phi : [0, \infty) \to [0, \infty) \) such that \( d(Tx, Ty) < \phi(d(x, y)) \)

\( \forall x \neq y \in X. \)

In (4) one can choose \( \phi \) to be also nondecreasing and right continuous.

Proof. (1) \( \Rightarrow \) (2): (1) implies that \( d(Tx, Ty) < d(x, y) \quad \forall x \neq y \in X. \) Thus Meir–Keeler’s condition (1) is valid even if \( d(x, y) < \varepsilon. \) Then by definition of \( \delta(\varepsilon) \) one has \( \delta(\varepsilon) > \varepsilon. \)

(2) \( \Rightarrow \) (3): If \( \delta(\varepsilon) < \infty, \) one can take \( \zeta = \delta \) and apply Proposition 3. If \( \delta(\varepsilon) = \infty \) for some \( \varepsilon > 0, \) let \( \varepsilon_0 = \inf \{ \varepsilon : \delta(\varepsilon) = \infty \}. \)

Case (i): \( \delta(\varepsilon_0) < \infty. \) Define \( \zeta(\varepsilon) = \delta(\varepsilon) \) for \( \varepsilon \leq \varepsilon_0 \) and \( \zeta(\varepsilon) = \delta(\varepsilon_0) + 2(\varepsilon - \varepsilon_0) \) for \( \varepsilon > \varepsilon_0. \)

Case (ii): \( \delta(\varepsilon_0) = \infty. \) Define \( \zeta(\varepsilon) = \delta(\varepsilon) \) for \( \varepsilon < \varepsilon_0, \zeta(\varepsilon_0) = 2\varepsilon_0, \) and \( \zeta(\varepsilon) = 2\varepsilon_0 + 2(\varepsilon - \varepsilon_0) \) for \( \varepsilon > \varepsilon_0. \)

Since \( \delta(\varepsilon) \) is nondecreasing and hence right lower semicontinuous, it is easy to see that in each case \( \zeta \) is right lower semicontinuous. By Proposition 3, \( \delta(d(Tx, Ty)) \leq d(x, y) \forall x, y \in X. \) Since the left-hand side of this inequality cannot be infinity, one has \( d(Tx, Ty) \leq \varepsilon_0 \forall x, y \in X \) in Case (i) and \( d(Tx, Ty) < \varepsilon_0 \forall x, y \in X \) in Case (ii). Therefore, \( \zeta(d(Tx, Ty)) = \delta(d(Tx, Ty)) \leq d(x, y) \forall x, y \in X. \)

(3) \( \Rightarrow \) (1): [4] Let \( \varepsilon > 0. \) By the right lower-semicontinuity of \( \zeta, \) there exists \( \lambda_1 > 0 \) such that

\[
\frac{\varepsilon + \zeta(\varepsilon)}{2} < \zeta(s) \quad \forall \varepsilon \leq s < \varepsilon + \lambda_1.
\]
Let \( \lambda = \min \{ \lambda_1, (\zeta(\varepsilon) - \varepsilon)/2 \} \). Condition (3) implies that \( \lambda > 0 \) and that \( d(Tx, Ty) < d(x, y) \) \( \forall x \neq y \in X \). Suppose \( \varepsilon \leq d(x, y) < \varepsilon + \lambda \). If \( d(Tx, Ty) \geq \varepsilon \), then \( \varepsilon \leq d(Tx, Ty) < d(x, y) < \varepsilon + \lambda \) and by (2)

\[
\frac{\varepsilon + \zeta(\varepsilon)}{2} < \zeta(d(Tx, Ty)) \leq d(x, y) < \varepsilon + \frac{\zeta(\varepsilon) - \varepsilon}{2} = \frac{\varepsilon + \zeta(\varepsilon)}{2},
\]

a contradiction. Hence, \( d(Tx, Ty) < \varepsilon \).

(2) \( \Rightarrow \) (4): Let \( \beta = \inf \{ \varepsilon : \delta(\varepsilon) = \infty \} \) (inf \( \emptyset = \infty \)). Let \( \varphi(\varepsilon) = (\varepsilon + \delta(\varepsilon))/2 \) and let \( \phi_0 \) and \( \psi \) be the pseudo-inverses of \( \varphi \) and \( \delta \), respectively. Obviously, \( \varphi(\beta +) = \delta(\beta +) = \infty \).

If \( \beta = 0 \), then \( T \) is a constant map and we may take \( \phi \) to be the zero function. So assume that \( \beta > 0 \).

First, we consider \( \varepsilon < \beta \). Clearly, \( \varphi(\varepsilon) < \delta(\varepsilon) \) \( \forall \beta > \varepsilon > 0 \). If \( \delta \) is continuous at 0, then for every \( \varepsilon > 0 \), there exists \( d > 0 \) such that \( \delta(s) \leq \varepsilon \forall s < d \); so by the definition of \( \psi \), one has \( \psi(\varepsilon) \leq d > 0 \) and hence \( \phi_0(\varepsilon) \geq \psi(\varepsilon) > 0 \). If \( \delta \) is discontinuous at 0, then so is \( \varphi \) and by item 3 in Proposition 1, \( \phi_0(t) = 0 = \psi(t) \forall t \in [0, \varphi(0+)) \).

Suppose \( \varepsilon > 0 \) and \( \delta \) is continuous at \( \varepsilon \). Let \( t = \varphi(\varepsilon) \). For every \( \varepsilon_1 > \varepsilon \), one has

\[
\varphi(\varepsilon_1) = \frac{\varepsilon_1 + \delta(\varepsilon_1)}{2} = \frac{\varepsilon + \delta(\varepsilon)}{2} = \varphi(\varepsilon),
\]

so \( \phi_0(t) = \varepsilon \) by the definition of pseudo-inverse. Since \( \delta(\varepsilon) > \varphi(\varepsilon) = t \), by the continuity of \( \delta \) at \( \varepsilon \), there exists \( \varepsilon_1 < \varepsilon \) such that \( \delta(\varepsilon_1) > t \). Hence, \( \psi(t) \leq \varepsilon_1 < \varepsilon = \phi_0(t) \).

Suppose \( \varepsilon > 0 \) and \( \delta \) is discontinuous at \( \varepsilon \). Then, \( \varphi \) is also discontinuous at \( \varepsilon \).

**Case (i):** \( \varphi(\varepsilon+) \leq \delta(\varepsilon-) \). Let \( t \in [\varphi(\varepsilon-), \varphi(\varepsilon+)) \). Then, \( t < \delta(\varepsilon-) \) and there exists \( \varepsilon_0 < \varepsilon \) such that \( \delta(s) > t \) \( \forall s > \varepsilon_0 \). Therefore, by the definition of \( \psi \), \( \psi(t) < \varepsilon \). By item 3 in Proposition 1, \( \phi_0(t) = \varepsilon \forall t \in [\varphi(\varepsilon-), \varphi(\varepsilon+)) \). Thus \( \phi_0(t) = \varepsilon > \psi(t) \forall t \in [\varphi(\varepsilon-), \varphi(\varepsilon+)) \).

**Case (ii):** \( \varphi(\varepsilon+) > \delta(\varepsilon-) \). Then on the interval \( [\varphi(\varepsilon-), \delta(\varepsilon-)) \), \( \phi_0(t) = \varepsilon > \psi(t) \) as in Case (i). Also note that \( \varphi(\varepsilon+) \leq \delta(\varepsilon+) \). Then by applying the item 3 in Proposition 1 to both \( \varphi \) and \( \delta \), we get \( \psi(t) = \varepsilon = \phi_0(t) \forall t \in [\delta(\varepsilon-), \varphi(\varepsilon+)) \).

The above paragraph is also valid if \( \beta < \infty \) and \( \varepsilon = \beta \).

To summarize, \( \phi_0(t) \geq \psi(t) \) and the equality holds only when \( t = 0 \) or in an interval of the form \( [\delta(\varepsilon-), \varphi(\varepsilon+)) \) for some \( \varepsilon \geq 0 \) where \( \delta \) is discontinuous (\( \delta(0-) \) is taken to be 0). Moreover, in the case of equality, \( \psi(t) = \phi_0(t) = \varepsilon \forall t \in [\delta(\varepsilon-), \varphi(\varepsilon+)) \).

Now we need to modify \( \phi_0 \) slightly to obtain \( \phi \).
If $\delta$ is discontinuous at 0, we define $\phi$ to be linear on the interval $[0, \varepsilon(0+))$ with $\phi(0) = 0$ and $\phi(\varepsilon(0+)) = \frac{1}{2}\varepsilon(0+)$. If $\delta$ is discontinuous at $\varepsilon > 0$ and $\delta(\varepsilon) > \varepsilon$ we define $\phi(t) = [\varepsilon + \delta(\varepsilon)]/2$ for $t \in [\delta(\varepsilon), \varepsilon(0+)]$. Suppose $\delta$ is discontinuous at $\varepsilon > 0$ and $\delta(\varepsilon) = \varepsilon$. We consider the following two cases.

Case (i): $\varepsilon(\varepsilon+) \leq \delta(\varepsilon)$; define $\phi(t) = \phi_0(t)$ on the interval $[\varepsilon, \varepsilon(\varepsilon+))$.

Case (ii): $\varepsilon(\varepsilon+) > \delta(\varepsilon)$; define $\phi(t) = \phi_0(t)$ on the interval $[\varepsilon, \delta(\varepsilon)]/2 \ \forall t \in [\delta(\varepsilon), \varepsilon(\varepsilon+))$. Define $\phi(t) = \phi_0(t)$ for any other $s$. The function $\phi$ so defined satisfies $\phi(t) > 0 \ \forall t > 0$, $\phi(t) \leq t$, $\phi(t) \geq \psi(t) \ \forall t \geq 0$ and the equality holds only when $t = 0$ or when $\delta(\varepsilon) = \varepsilon$, $\phi(t) = \psi(t) = \varepsilon \ \forall t \in [\varepsilon, \min\{\delta(\varepsilon), \varepsilon(\varepsilon+)\}]$.

Since $\phi_0$ is right continuous by Proposition 1, it is easy to see that $\phi$ is also right continuous. It follows from the proof of Proposition 5 that $\phi$ is an $L$-function. By Proposition 2, $d(Tx, Ty) \leq \psi(d(x, y)) < \phi(d(x, y)) \ \forall x, y \in X$ except when $d(x, y) = 0$ or $d(x, y) \in [\varepsilon, \min\{\delta(\varepsilon), \varepsilon(\varepsilon+)\}]$ for some $\varepsilon > 0$ such that $\delta(\varepsilon) = \varepsilon$. In the latter case, since $d(x, y) < \delta(\varepsilon)$, we have $d(Tx, Ty) < \varepsilon = \phi(d(x, y))$. This completes the proof that (2) implies (4).

The function $\phi$ above may not be nondecreasing. If we define $\zeta(s) = \sup\{\phi(t): t \leq s\}$, then $\zeta$ is a nondecreasing, right continuous $L$-function that can be used in place of $\phi$.

(4) $\Rightarrow$ (1): Suppose $d(Tx, Ty) < \phi(d(x, y))$ for $x \neq y \in X$, for some $L$-function $\phi$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\phi(t) \leq \varepsilon \ \forall t \in [\varepsilon, \varepsilon + \delta)$. So if $\varepsilon \leq d(x, y) < \varepsilon + \delta$, then $d(Tx, Ty) < \phi(d(x, y)) \leq \varepsilon$. □

**Remark 1.** It is easy to see that the function $\phi$ in the Boyd–Wong’s theorem is an $L$-function. However, the inequality there is not strict. To get strict inequality as in item 4, simply replace the function $\phi(s)$ there by $[s + \phi(s)]/2$.

**Remark 2.** It is not necessary to require that $\phi(s) > 0 \ \forall s > 0$ in Theorem 1 if one drops the condition from the definition of $L$-function and change the condition

$$d(Tx, Ty) < \phi(d(x, y)) \ \forall x \neq y \in X$$

to

$$d(Tx, Ty) < \phi(d(x, y)), \ \forall x, y \in X \text{ with } \phi(d(x, y)) > 0.$$  

**Remark 3.** Let $X = [0, 1] \cup \{3n, 3n + 1\}_{n=1}^{\infty}$ with the Euclidean metric and $T: X \rightarrow X$ the map defined by [1]

$$T(x) = \begin{cases} 
\frac{x}{2} & \text{for } 0 \leq x \leq 1, \\
0 & \text{for } x = 3n, \\
1 - \frac{x}{n+2} & \text{for } x = 3n + 1.
\end{cases}$$

Then,

$$\delta(\varepsilon) = \begin{cases} 
2\varepsilon & \text{for } 0 \leq \varepsilon \leq 1/2, \\
1 & \text{for } 1/2 < \varepsilon < 1, \\
\infty & \text{for } 1 \leq \varepsilon < \infty.
\end{cases}$$
\[ \psi(t) = \begin{cases} 
\frac{t}{2} & \text{for } 0 \leq t < 1, \\
1 & \text{for } 1 \leq t < \infty 
\end{cases} \]

and

\[ \phi(t) = \phi_0(t) = \begin{cases} 
\frac{2}{3}t & \text{for } 0 \leq t \leq 3/4, \\
2t - 1 & \text{for } 3/4 < t \leq 1, \\
1 & \text{for } 1 \leq t < \infty. 
\end{cases} \]

Note that \( \phi(1) = 1. \) Indeed, for any \( L \)-function \( \phi \) satisfying \( d(Tx, Ty) < \phi(d(x, y)) \) \( \forall x \neq y \in X, \) we have, by setting \( x = 3n, \ y = 3n + 1, \ 1 - 1/(n + 2) < \phi(1). \) Thus \( 1 \leq \phi(1). \) But \( 1 \geq \phi(1) \) by definition. Hence, \( \phi(1) = 1. \) This shows that it is not always possible to find a \( \phi \) such that \( \phi^n \) converges pointwise to 0.

Let \((X, d)\) be a metric space and \( T : X \to 2^X \setminus \{\emptyset\} \) a multivalued map with \( Tx \) closed for every \( x \in X. \) Let \( H \) be the Hausdorff metric on nonempty subsets of \( H, \) i.e.

\[ H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(a, A) \right\}, \]

where \( d(x, A) = \inf_{a \in A} d(x, a). \) Meir–Keeler’s condition and modulus of uniform continuity of \( T \) can be defined similarly:

\[ \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{such that } \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow H(Tx, Ty) < \varepsilon, \] (3)

\[ \delta(\varepsilon) = \inf \{ d(x, y) : H(Tx, Ty) \geq \varepsilon \}. \] (4)

The following theorem remains valid.

**Theorem 2.** Let \( X \) be a metric space. Let \( T : X \to 2^X \setminus \{\emptyset\} \) and let \( \delta(\varepsilon) \) be its modulus of uniform continuity. The following are equivalent:

1. \( T \) satisfies condition (3).
2. \( \delta(\varepsilon) > \varepsilon \ \forall \varepsilon > 0. \)
3. There exists a right lower semicontinuous function \( \zeta : [0, \infty) \to [0, \infty) \) such that \( \zeta(0) = 0, \zeta(\varepsilon) > \varepsilon \) for every \( \varepsilon > 0 \) and \( \zeta(H(Tx, Ty)) \leq d(x, y) \) for every \( x, y \in X. \)
4. There exists an \( L \)-function \( \phi : [0, \infty) \to [0, \infty) \) such that \( H(Tx, Ty) < \phi(d(x, y)) \) \( \forall x \neq y \in X. \)

In (4) one can choose \( \phi \) to be also nondecreasing and right continuous.

**Open Problem.** Let \((X, d)\) be a complete metric space and \( T : X \to 2^X \setminus \{\emptyset\} \) a multivalued mapping such that \( Tx \) is closed for every \( x \) and

\[ H(Tx, Ty) < \psi(d(x, y)), \ \forall x \neq y \in X, \]

where \( H \) denotes the Hausdorff metric and \( \psi \) is an \( L \)-function. Does \( T \) have a fixed point?

The answer is yes if \( Tx \) is compact for every \( x \) [3].
References