Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”

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ABSTRACT

Huang and Zhang reviewed cone metric spaces in 2007 [Huang Long-Guang, Zhang Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476]. We shall prove that there are no normal cones with normal constant \( M < 1 \) and for each \( k > 1 \) there are cones with normal constant \( M > k \). Also, by providing non-normal cones and omitting the assumption of normality in some results of [Huang Long-Guang, Zhang Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476], we obtain generalizations of the results.

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1. Introduction

Recently, non-convex analysis has found some applications in optimization theory, and so there have been some investigations about non-convex analysis, especially ordered normed spaces, normal cones and Topical functions (for example [1–3]). In these efforts an order is introduced by using vector space cones. Huang and Zhang used this approach in [1]; they defined cone metric spaces by substituting an ordered normed space for the real numbers. In this paper, we shall show that there are no normal cones with normal constant \( M < 1 \), and for each \( k > 1 \) there are cones with normal constant \( M > k \). Also, by providing non-normal cones and omitting the assumption of normality in some results of [1], we obtain generalizations of the results.

Let \( E \) always be a real Banach space and \( P \) a subset of \( E \). \( P \) is called a cone if

(i) \( P \) is closed, non-empty and \( P \neq \{0\} \),
(ii) \( ax + by \in P \) for all \( x, y \in P \) and non-negative real numbers \( a, b \),
(iii) \( P \cap (-P) = \{0\} \).

For a given cone \( P \subseteq E \), we can define a partial ordering \( \leq \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). \( x < y \) will stand for \( x \leq y \) and \( x \neq y \), while \( x \leq y \) will stand for \( y - x \in \text{int}\ P \), where \( \text{int}\ P \) denotes the interior of \( P \).

The cone \( P \) is called normal if there is a number \( M > 0 \) such that for all \( x, y \in E \),

\[
0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq M\|y\|.
\]

The least positive number satisfying the above is called the normal constant of \( P \) [1].

The cone \( P \) is called regular if every increasing sequence which is bounded from above is convergent. That is, if \( \{x_n\}_{n \geq 1} \) is a sequence such that \( x_1 \leq x_2 \leq \cdots \leq y \) for some \( y \in E \), then there is \( x \in E \) such that \( \lim_{n \to \infty} \|x_n - x\| = 0 \). Equivalently

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the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been mentioned that every regular cone is normal [1], but we prove it here.

**Lemma 1.1.** Every regular cone is normal.

**Proof.** Let $P$ a regular cone which is not normal. For each $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2 \|t_n\| < \|s_n\|$. For each $n \geq 1$, put $y_n = \frac{t_n}{\|s_n\|}$ and $x_n = \frac{s_n}{\|s_n\|}$. Then, $x_n, y_n, y_n - x_n \in P$, $\|y_n\| = 1$ and $n^2 < \|x_n\|$, for all $n \geq 1$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \|y_n\|$ is convergent and $P$ is closed, there is an element $y \in P$ such that $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$. Now, note that

$$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \leq \cdots \leq y.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is convergent because $P$ is regular. Hence,

$$\lim_{n \to \infty} \frac{\|x_n\|}{n^2} = 0,$$

which is a contradiction. □

In the following we always suppose that $E$ is a Banach space, $P$ is a cone in $E$ with $int P \neq \emptyset$ and $\leq$ is partial ordering with respect to $P$.

**Definition 1.1.** Let $X$ be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space [1].

**Example 1.2.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space [1].

**Definition 1.3.** (See [1].) Let $(X, d)$ be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in $X$. Then

1. $\{x_n\}_{n \geq 1}$ converges to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. $(X, d)$ is a complete cone metric space if every Cauchy sequence is convergent.

Finally, note that the relations $int P + int P \subseteq int P$ and $\lambda int P \subseteq int P$ ($\lambda > 0$) hold. We appeal to these relations in the following.

2. Main results

First, we review types of cones. We then prove that there are no cones with normal constant $M < 1$. And we show by example that there exist cones of normal constant 1, and cones of normal constant $M > k$ for each $k > 1$.

**Lemma 2.1.** There is not normal cone with normal constant $M < 1$.

**Proof.** Let $(X, d)$ be a cone metric space and $P$ a normal cone with normal constant $M < 1$. Choose a non-zero element $x \in P$ and $0 < \varepsilon < 1$ such that $M < 1 - \varepsilon$. Then, $(1 - \varepsilon)x \leq x$, but $(1 - \varepsilon)\|x\| > M\|x\|$. This is a contradiction. □

Now by using Lemma 2.1, we can provide normal cones with normal constant 1.

**Example 2.1.** Let $E = C_b([0, 1])$ with the supremum norm and

$$P = \{f \in E : f \geq 0\}.$$

Then, $P$ is a cone with normal constant of $M = 1$. Now, consider the following sequence of elements of $E$ which is decreasing and bounded from below but it is not convergent in $E$. 
\[ x > x^2 > x^3 > \cdots > 0. \]

Therefore, the converse of Lemma 1.1 is not true.

**Example 2.2.** Let \( E = \ell^1 \), \( P = \{ (x_n)_{n \geq 1} \in E : x_n \geq 0 \} \), for all \( n \), \((X, \rho)\) a metric space and \( d : X \times X \to E \) defined by \( d(x, y) = \{ \frac{\rho(x, y)}{2^m} \}_{m \geq 1} \). Then \((X, d)\) is a cone metric space and the normal constant of \( P \) is equal to \( M = 1 \).

Moreover, this example shows that the category of cone metric spaces is bigger than the category of metric spaces.

**Proposition 2.2.** For each \( k > 1 \), there is a normal cone with normal constant \( M > k \).

**Proof.** Let \( k > 1 \) be given. Consider the real vector space
\[ E = \{ ax + b \mid a, b \in \mathbb{R}; \ x \in \left[ 1 - \frac{1}{k}, 1 \right] \}, \]
with supremum norm and the cone
\[ P = \{ ax + b \in E \mid a \leq 0, \ b \geq 0 \} \]
in \( E \). First, we show that \( P \) is regular (and so normal). Let \( \{ a_n x + b_n \}_{n \geq 1} \) be an increasing sequence which is bounded from above, that is, there is an element \( c x + d \in E \) such that
\[ a_1 x + b_1 \leq a_2 x + b_2 \leq \cdots \leq a_n x + b_n \leq \cdots \leq c x + d, \]
for all \( x \in [1 - \frac{1}{k}, 1] \). Then, \( \{ a_n x \}_{n \geq 1} \) and \( \{ b_n \}_{n \geq 1} \) are two sequences in \( \mathbb{R} \) such that
\[ b_1 \leq b_2 \leq \cdots \leq d, \quad a_1 > a_2 > \cdots \geq c. \]
Thus, \( \{ a_n \}_{n \geq 1} \) and \( \{ b_n \}_{n \geq 1} \) are convergent. Let \( a_n \to a \) and \( b_n \to b \). Then, \( ax + b \in P \) and \( a_n x + b_n \to ax + b \). Therefore, \( P \) is regular. Hence by Lemma 2.1, there is \( M > 1 \) such that \( 0 \leq g \leq f \) implies \( \|g\| \leq M\|f\| \), for all \( g, f \in E \). Now, we show that \( M > k \). First, note that \( f(x) = -k x + k \in P \), \( g(x) = k \in P \) and \( f - g \in P \). So, \( 0 \leq g \leq f \). Therefore, \( k = \|g\| \leq M\|f\| = M \).

On the other hand, if we consider \( f(x) = -(k + \frac{1}{k}) x + k \) and \( g(x) = k \), then \( f \in P \), \( g \in P \) and \( f - g \in P \). Also, \( \|g\| = k \) and \( \|f\| = 1 - \frac{1}{k} + \frac{1}{k^2} \). Thus, \( k = \|g\| > \|f\| = k + \frac{1}{k} - 1 \). This shows that \( M > k \). \( \square \)

The following example shows that there are non-normal cones.

**Example 2.3.** Let \( E = C_c^\infty([0, 1]) \) with the norm
\[ \|f\| = \|f\|_\infty + \|f'\|_\infty, \]
and consider the cone
\[ P = \{ f \in E : f \geq 0 \}. \]
For each \( k \geq 1 \), put \( f(x) = x \) and \( g(x) = x^{2k} \). Then, \( 0 \leq g \leq f \), \( \|f\| = 2 \) and \( \|g\| = 2k + 1 \). Since \( k\|f\| < \|g\| \), \( k \) is not normal constant of \( P \). Therefore, \( P \) is a non-normal cone.

Now, we generalize some results of [1] by omitting the assumption of normality in the results.

**Theorem 2.3.** Let \((X, d)\) be a complete cone metric space and the mapping \( T : X \to X \) satisfy the contractive condition
\[ d(Tx, Ty) \leq kd(x, y), \]
for all \( x, y \in X \), where \( k \in (0, 1) \) is a constant. Then, \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{ T^n x \}_{n \geq 1} \) converges to the fixed point.

**Proof.** For each \( x_0 \in X \) and \( n \geq 1 \), set \( x_1 = Tx_0 \) and \( x_{n+1} = Tx_n = T^{n+1}x_0 \). Then,
\[ d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}) \leq k^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq k^n d(x_1, x_0). \]
So for \( n > m \),
\[ d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq (k^{n-1} + k^{n-2} + \cdots + k^m) d(x_1, x_0) \leq \frac{k^m}{1-k} d(x_1, x_0). \]
Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{ y \in E : \| y \| < \delta \}$. Also, choose a natural number $N_1$ such that $\frac{k^n}{1-k}d(x_1, x_0) \ll N_1$, for all $m \geq N_1$. Then, $\frac{k^n}{1-k}d(x_1, x_0) \ll c$, for all $m \geq N_1$. Thus,

$$d(x_n, x_m) \ll \frac{p_m}{1-k}d(x_1, x_0) \ll c,$$

for all $n > m$. Therefore, $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone metric space, there exists $x^* \in X$ such that $x_n \rightarrow x^*$. Choose a natural number $N_2$ such that $d(x_n, x^*) \ll \frac{c}{2}$ for all $n \geq N_2$. Hence,

$$d(Tx^*, x^*) \leq d(Tx_1, Tx^*) + d(Tx^*, x^*) \leq kd(x_n, x^*) + d(x_{n+1}, x^*) \leq d(x_n, x^*) + d(x_{n+1}, x^*) \ll \frac{c}{2} + \frac{c}{2} = c,$$

for all $n \geq N_2$. Thus, $d(Tx^*, x^*) \ll \frac{c}{2}$, for all $n \geq N_2$. Since $\frac{c}{2m} \rightarrow 0$ (as $m \rightarrow \infty$) and $P$ is closed, $-d(Tx^*, x^*) \in P$. But, $d(Tx^*, x^*) \in P$. Therefore, $d(Tx^*, x^*) = 0$, and so $Tx^* = x^*$. $\square$

Thus, we can generalize Corollaries 1 and 2 of [1] as follows.

**Corollary 2.4.** Let $(X, d)$ be a complete cone metric space. For $0 \ll c$ and $x_0 \in X$, set $B(x_0, c) = \{ x \in X : d(x_0, x) \ll c \}$. Suppose that the mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in B(x_0, c)$, where $k \in [0, 1)$ is a constant and $d(Tx_0, x_0) \leq (1-k)c$. Then, $T$ has a unique fixed point in $B(x_0, c)$.

**Corollary 2.5.** Let $(X, d)$ be a complete cone metric space. Suppose a mapping $T : X \rightarrow X$ satisfies for some positive integer $n$,

$$d(T^n x, T^n y) \leq kd(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then, $T$ has a unique fixed point in $X$.

**Theorem 2.6.** Let $(X, d)$ be a complete cone metric space and the mapping $T : X \rightarrow X$ satisfy the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)),
$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then, $T$ has a unique fixed point in $X$. For each $x \in X$, the iterative sequence $(T^n x)_{n \geq 1}$ converges to the fixed point.

**Proof.** For each $x_0 \in X$ and $n \geq 1$, set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq k(d(x_n, x_n) + d(Tx_{n-1}, x_{n-1})) = k(d(x_{n+1}, x_n) + d(x_n, x_{n-1})).$$

So,

$$d(x_{n+1}, x_n) \leq \frac{k}{1-k}d(x_n, x_{n-1}) = hd(x_n, x_{n-1}),$$

where $h = \frac{k}{1-k}$. For $n > m$,

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq (h^{n-1} + h^{n-2} + \cdots + h^m)d(x_1, x_0) \leq \frac{h^m}{1-h}d(x_1, x_0).$$

Let $0 \ll c$ be given. Choose a natural number $N_1$ such that $\frac{h^m}{1-h}d(x_1, x_0) \ll c$, for all $m \geq N_1$. Thus,

$$d(x_n, x_m) \ll c,$$

for $n > m$. Therefore, $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone metric space, there exists $x^* \in X$ such that $x_n \rightarrow x^*$. Choose a natural number $N_2$ such that $d(x_{n+1}, x_n) \ll \frac{c(1-k)}{2k}$ and $d(x_{n+1}, x^*) \ll \frac{c(1-k)}{2}$, for all $n \geq N_2$. Hence, for $n \geq N_2$ we have

$$d(Tx^*, x^*) \leq d(Tx_1, Tx^*) + d(Tx^*, x^*) \leq k(d(Tx_1, x_1) + d(Tx^*, x^*)) + d(x_{n+1}, x^*).$$

Thus,

$$d(Tx^*, x^*) \leq \frac{1}{1-k}(kd(x_{n+1}, x_n) + d(x_{n+1}, x^*)) \ll \frac{c}{2} + \frac{c}{2} = c.$$
Thus, \( d(Tx^n, x^*) \ll \frac{c}{m} \), for all \( m \geq 1 \). So, \( \frac{c}{m} - d(Tx^n, x^*) \in P, \) for all \( m \geq 1 \). Since \( \frac{c}{m} \to 0 \) (as \( m \to \infty \)) and \( P \) is closed, \(-d(Tx^n, x^*) \in P \). But, \( d(Tx^n, x^*) \in P \). Therefore, \( d(Tx^n, x^*) = 0 \), and so \( Tx^n = x^* \). Now, if \( y^* \) is another fixed point of \( T \), then
\[
d(x^*, y^*) = d(Tx^*, Ty^*) \leq k(d(Tx^n, x^*) + d(Ty^n, y^*)) = 0.
\]
Hence, \( x^* = y^* \). Therefore, the fixed point of \( T \) is unique. \( \square \)

**Theorem 2.7.** Let \((X, d)\) be a complete cone metric space and the mapping \( T : X \to X \) satisfy the contractive condition
\[
d(Tx, Ty) \leq k(d(x, y) + d(x, Tx)),
\]
for all \( x, y \in X \), where \( k \in [0, \frac{1}{2}) \) is a constant. Then, \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{T^n x\}_{n \geq 1} \) converges to the fixed point.

**Proof.** For each \( x_0 \in X \) and \( n \geq 1 \), set \( x_1 = Tx_0 \) and \( x_{n+1} = Tx_n = T^{n+1} x_0 \). Then,
\[
d(x_{n+1}, x_n) = d(Tx_n, T^{n+1} x_0) = k(d(Tx_n, x_n) + d(Tx_0, x_n)).
\]
So,
\[
d(x_{n+1}, x_n) \leq k \frac{d(x_n, x_{n-1})}{1-k} = k d(x_n, x_{n-1}),
\]
where \( h = \frac{k}{1-k} \). For \( n > m \),
\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1, x_0) \leq \frac{h^m}{1-h} d(x_1, x_0).
\]
Let \( 0 \ll c \) be given. Choose a natural number \( N_1 \) such that \( \frac{h^m}{1-h} d(x_1, x_0) \ll c \), for all \( m \geq N_1 \). Thus,
\[
d(x_m, x_0) \ll c,
\]
for \( n > m \). Therefore, \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence in \((X, d)\). Since \((X, d)\) is a complete cone metric space, there exists \( x^* \in X \) such that \( x_n \to x^* \). Choose a natural number \( N_2 \) such that \( d(x_n, x^*) \ll \frac{c}{1+k} \), for all \( n \geq N_2 \). Hence, for \( n \geq N_2 \) we have
\[
d(Tx^n, x^*) \leq d(Tx^n, x^*) + d(Tx_n, x^*) \leq k(d(Tx^n, x^*) + d(Tx_0, x^*) + d(x_{n+1}, x^*)
\]
\[
\leq k(d(Tx^n, x^*) + d(x_n, x^*) + d(x_{n+1}, x^*) + d(x_{n+1}, x^*)).
\]
Thus,
\[
d(Tx^n, x^*) \leq \frac{1}{1-k} (k d(x_n, x^*) + d(x_{n+1}, x^*) + d(x_{n+1}, x^*)) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.
\]
Thus, \( d(Tx^n, x^*) \ll \frac{c}{m} \), for all \( m \geq 1 \). So, \( \frac{c}{m} - d(Tx^n, x^*) \in P, \) for all \( m \geq 1 \). Since \( \frac{c}{m} \to 0 \) (as \( m \to \infty \)) and \( P \) is closed, \(-d(Tx^n, x^*) \in P \). But, \( d(Tx^n, x^*) \in P \). Therefore, \( d(Tx^n, x^*) = 0 \), and so \( Tx^n = x^* \). Now, if \( y^* \) is another fixed point of \( T \), then
\[
d(x^*, y^*) = d(Tx^*, Ty^*) \leq k(d(Tx^n, y^*) + d(Ty^n, x^*)) = 2kd(x^*, y^*).
\]
Hence, \( d(x^*, y^*) = 0 \) and so \( x^* = y^* \). Therefore, the fixed point of \( T \) is unique. \( \square \)

Finally we provide another result which is not proved in [1].

**Theorem 2.8.** Let \((X, d)\) be a complete cone metric space and the mapping \( T : X \to X \) satisfy the contractive condition
\[
d(Tx, Ty) \leq k(d(x, y) + d(y, Tx)),
\]
for all \( x, y \in X \), where \( k, l \in [0, 1) \) is a constant. Then, \( T \) has a fixed point in \( X \). Also, the fixed point of \( T \) is unique whenever \( k + l < 1 \).

**Proof.** For each \( x_0 \in X \) and \( n \geq 1 \), set \( x_1 = Tx_0 \) and \( x_{n+1} = Tx_0 = T^{n+1} x_0 \). Then,
\[
d(x_{n+1}, x_n) = d(Tx_n, T^{n+1} x_0) = k(d(x_n, x_{n-1}) + d(Tx_0, x_n)) = k d(x_n, x_{n-1}) \leq k d(x_1, x_0).
\]
Thus for \( n > m \), we have
\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq (k^{n-1} + k^{n-2} + \cdots + k^m) d(x_1, x_0) \leq \frac{k^m}{1-k} d(x_1, x_0).
\]
Let \( 0 \ll c \) be given. Choose a natural number \( N_1 \) such that \( \frac{km}{k+1}d(x_1, x_0) \ll c \), for all \( m \geq N_1 \). Thus,
\[
d(x_n, x_m) \ll c,
\]
for \( n > m \). Therefore, \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence in \((X, d)\). Since \((X, d)\) is a complete cone metric space, there exists \( x^* \in X \) such that \( x_n \to x^* \). Choose a natural number \( N_2 \) such that \( d(x_n, x^*) \ll \frac{c}{3} \), for all \( n \geq N_2 \). Hence, for \( n > N_2 \) we have
\[
d(Tx^*, x^*) \leq d(x_n, Tx^*) + d(x_n, x^*) = d(x_n-1, Tx^*) + d(x_n, x^*) \leq kd(x_{n-1}, x^*) + ld(Tx_{n-1}, x^*) + d(x_n, x^*)
\]
\[
\leq d(x_{n-1}, x^*) + d(x_n, x^*) + d(x_n, x^*) \leq \frac{1}{3}c + \frac{1}{3}c + \frac{1}{3}c = c.
\]
So,
\[
d(Tx^*, x^*) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.
\]
Thus, \( d(Tx^*, x^*) \ll \frac{c}{m} \), for all \( m \geq 1 \). Hence, \( \frac{c}{m} - d(Tx^*, x^*) \in P \), for all \( m \geq 1 \). Since \( \frac{c}{m} \to 0 \) (as \( m \to \infty \)) and \( P \) is closed, \(-d(Tx^*, x^*) \in P \). But, \( d(Tx^*, x^*) \in P \). Therefore, \( d(Tx^*, x^*) = 0 \), and so \( Tx^* = x^* \). Now, if \( y^* \) is another fixed point of \( T \) and \( k + l < 1 \), then
\[
d(x^*, y^*) = d(Tx^*, y^*) \leq kd(x^*, y^*) + ld(Tx^*, y^*) = (k + l)d(x^*, y^*)
\]
Hence, \( d(x^*, y^*) = 0 \) and so \( x^* = y^* \). Therefore, the fixed point of \( T \) is unique whenever \( k + l < 1 \). \( \square \)

References