Series I, exercise 1 Show that there are infinitely many sequences of 23 consecutive natural numbers such that sums of squares of all their terms are equal to squares of natural numbers.

Solution We need to show that the equation

$$(x-11)^2 + (x-10)^2 + \dots + (x-1)^2 + x^2 + (x+1)^2 + \dots (x+11)^2 = y^2$$

has infinitely many solutions in natural numbers, where $x > 11$. We have an equivalent equation:

$$23x^2 + x(-22 - 20 - \dots + 20 + 22) + (-11)^2 + (-10)^2 + \dots + 11^2 = y^2$$

and further

$$23x^2 + 2(1^2 + 2^2 + \dots + 11^2) = y^2,$$

so

$$23x^2 + 1012 = y^2. \tag{1}$$

Moreover, $1012 = 23 \cdot 44$, and hence $23 | y^2$, so $23 | y$. Therefore, $y = 23t$ for $t \in \mathbb{N}$. Dividing both sides of equation (1) by 23 we obtain

$$x^2 + 44 = 23t^2. \tag{2}$$

We are looking for t such that $f(t) := 23t^2 - 44$ is a square of some natural number. We have $f(1) = -21$, $f(2) = 48$, $f(3) = 163$,

$$f(4) = 324 = 18^2.$$

So, for $t_1 = 4$ ($y_1 = 23t_1 = 92$) and $x_1 = 18$ we have one solution. Now, we would like to find new pair (x_2, t_2) satisfying equation (2), which will depend, in some way, on x_1 and t_1. More generally, we would like to find a sequence of pairs (x_n, t_n) satisfying equation (2) and which can be described using (x_{n-1}, t_{n-1}). The most simple dependence is linear.

Assume that (x, t) satisfies equation (2). We are looking for $a, b, c, d \in \mathbb{N}$ such that pair (x', t') satisfies equation (2), where $x' = ax + bt$ and $t' = cx + dt$. So, we want to have

$$x'^2 - 23t'^2 = -44 = x^2 - 23t^2. \tag{3}$$

Then we will be able to define a sequence $((x_n, t_n))$ of pairs satisfying (2). We have

$$x'^2 - 23t'^2 = (ax + bt)^2 - 23(cx + dt)^2 = a^2x^2 + 2abxt + b^2t^2 - 23c^2x^2 - 46cdxt - 23d^2t^2$$

$$= x^2(a^2 - 23c^2) + xt(2ab - 46cd) + t^2(b^2 - 23d^2).$$

So, equation (3) is equivalent to

$$x^2(a^2 - 23c^2) + xt(2ab - 46cd) + t^2(b^2 - 23d^2) = x^2 - 23t^2,$$

which is satisfied if

$$\begin{cases} a^2 - 23c^2 = 1 \\ 2ab - 46cd = 0 \\ b^2 - 23d^2 = -23. \end{cases} \tag{4}$$

The third equation is equivalent to

$$b^2 = 23(d^2 - 1) = 23(d - 1)(d + 1).$$

Since $23(d-1)(d+1)$ must be a square of a natural number, one of $(d-1)$ or $(d+1)$ must be divisible by 23. The least such d is 24. Then we have

$$b^2 = 23 \cdot 23 \cdot 25 = 23^2 \cdot 5^2 = 105^2.$$

Hence $b = 105$ and $d = 24$ satisfy this equation. Now, putting these values to the second equation in (4) we obtain

$$a = c \cdot \frac{23 \cdot 24}{105} = c \cdot \frac{24}{5}.$$

This is satisfied for $c = 5$ and $a = 24$. We check, if then the first equation in (4) is satisfied. We have

$$a^2 - 23c^2 = 24^2 - 23 \cdot 25 = 576 - 575 = 1.$$

So, we proved that if (x, t) satisfies the equation (4), then the pair $(24x + 105t, 5x + 24t)$ also satisfies it. Now, we can recursively define a sequence $((x_n, t_n))$ satisfying the equation (3), putting $x_1 = 18, t_1 = 4$ and $x_n = 24x_{n-1} + 105t_{n-1}$, $t_n = 5x_{n-1} + 24t_{n-1}$ for $n \geq 2$. Of course, this sequence is increasing (with respect to both terms), and so it is infinite. Thus, we have the assertion.

Series I, exercise 2 A group of $2n$ students wrote a test on which the possible scores were $0, 1, \ldots, 10$. Each of these marks occurred at least once, and the average score was equal to $7, 4$. Prove that it is possible to divide the group into two subgroups of n students in such a way that the average score for each group was also equal to $7, 4$.

Solution Let S denote the sum of results obtained by the students. As $S = (2N)(7.4) = \frac{74n}{5}$ we have that n is divisible by 5 and S is even, thus divisible by 10. The number of students is also divisible by 10. Let us denote the scores of the students by $s_1, \ldots, s_{10m}$ where $s_1 + \cdots + s_{10m} = 74m$. If we can rearrange the scores such that $s_1 + \cdots + s_{5m} = 37m$ we are done. If not we can rearrange them so $S_1 = s_1 + \cdots + s_{5m} = 37m - \delta$ and $S_2 = s_{5m+1} + \cdots + s_{10m} = 37m + \delta$ with $\delta > 0$ as small as possible. If there is a term in S_1 that is exactly one less than some term in S_2 we can exchange these terms and make δ smaller, a contradiction. So, because all scores between 0 and 10 occur, if the smallest term in S_1 is a then $a + 1$ must also appear in S_1 and so $a + 2$ etc. must also appear in S_1. Let the largest therm in S_2 be b. Then $5ma \leq S_1 < S_2 \leq 5mb$, so $b > a$, so $b - 1$ must appear in S_1 and we can exchange $b - 1$ from S_1 and b from S_2 to reduce δ, a contradiction.

Series I, exercise 3 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(xy) = \frac{f(x) + f(y)}{x + y}.$$

for all $x, y \in \mathbb{R}$, with $x \neq -y$. Is there $x \in \mathbb{R}$ such that $f(x) \neq 0$.

Solution For $y = 0$ we have

$$f(0) = \frac{f(x) + f(0)}{x} \; (x \neq 0).$$

From this equation we obtain $f(x) = f(0)(x - 1)$ and for $x = 1$ we get $f(1) = 0$.

For $y = 1$ we have

$$f(x) = \frac{f(x) + f(1)}{x + 1} \; (x \neq -1).$$

Hence we obtain $xf(x) = 0$ and we have $f(x) = 0$ for all $x \neq 0, -1$. Now if we put $x = 2$, $y = 0$ and we get $f(0) = 0$ and for $x = 0$, $y = -1$ we obtain $f(-1) = 0$.