Series V, exercise 1 Find all pairs (p, q) of prime numbers such that there exist $n, k \in \mathbb{N}$, $k \geq 2$ for which $(p+1)^q - 1 = n^k$.

Solution Suppose that a pair of prime numbers $p, q \in \mathbb{N}$ are such that for some natural $n, k \geq 2$ we have

$$(p+1)^q - 1 = n^k.$$

Then, by Newton's formula, we have

$$\sum_{i=0}^{q} \binom{q}{i} p^i - 1 = n^k,$$

so

$$\sum_{i=2}^{q} \binom{q}{i} p^i + qp = n^k.$$

Thus, $p|n^k$. Since p is a prime number, we have $p|n$, and hence $n = bp$ for some $b \in \mathbb{N}$. Therefore,

$$\sum_{i=2}^{q} \binom{q}{i} p^i + qp = b^k p^k.$$

Dividing the above equality by p we receive

$$\sum_{i=1}^{q-1} \binom{q}{i} p^i + q = b^k p^{k-1}.$$

Since $k \geq 2$, we have $q|p$ and because both p and q are prime numbers, they must be equal. Now, we have

$$\sum_{i=2}^{q} \binom{q}{i} q^i + q^2 = b^k q^k.$$

Suppose that $q > 2$ and $k > 2$. Then

$$\frac{q(q-1)}{2} \cdot q^2 + \sum_{i=3}^{q} \binom{q}{i} q^i + q^2 = b^k q^k,$$

and so

$$\frac{q(q-1)}{2} + \sum_{i=1}^{q-2} \binom{q}{i} q^i + 1 = b^k q^{k-2}$$

Since q is odd, $\frac{q(q-1)}{2}$ is divisible by q, but $q \not| 1$ a contradiction. Thus $q = 2$ or $k = 2$.

If $q = 2$, then $p = 2$ and we have

$$(p+1)^q - 1 = 8 = 2^3,$$

so the pair $(2, 2)$ is "good".

If $k = 2$ and $q > 2$, then for some natural l

$$(p+1)^q - 1 = (q+1)^q - 1 = (2l+1)^2,$$

because q is odd. Hence

$$(q+1)^q = 2(2l^2 + 2l + 1),$$

so $2|(q+1)^q$ and $4 \not| (q+1)^q$, which is obviously not possible. Thus, the pair $(2, 2)$ is the only solution of our problem.

Series V, exercise 2 Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the following equality for all $x, y \in \mathbb{R}$

$$f(x+2y) = 2f(x)f(y).$$

Solution

For $y = 0$ we have $f(x) = 2f(x)f(0)$ for every $x \in \mathbb{R}$.

If $f(0) = \frac{1}{2}$, then for $x = 0$ we get $f(2y) = f(y)$ for every $y \in \mathbb{R}$. Hence for $a \in \mathbb{R}$ we obtain $f(a) = f\left(\frac{a}{2^n}\right)$ for all $n \in \mathbb{N}$. Therefore, by continuity of f we obtain that

$$f(a) = \lim_{n\to\infty} f\left(\frac{a}{2^n}\right) = f\left(\lim_{n\to\infty} \frac{a}{2^n}\right) = f(0)$$

It means $f(x) = \frac{1}{2}$ for every $x \in \mathbb{R}$.

If $f(0) \neq \frac{1}{2}$, then we get $f(x) = 0$ for every $x \in \mathbb{R}$ and f is a constant function.

Series V, exercise 3 Let $n, k \in \mathbb{N}$, $n > k$. Players A and B play a game with n pawns and a board consisting of k squares in one line. At the beginning of the game, the pawns are placed on k leftmost squares. In each turn a player can move any pawn to any free square which is further to the right. The players alternate turns, with player A starting the game. The game ends when the player cannot move (all pawns are on the rightmost squares), and so loses the game. For what n and k player A has a winning strategy, that is, can plan his moves (depending on the moves of his rival) in such a way that he can be sure at the start that he will win no matter of the moves of the rival.

Solution

Number the squares, from left to right, $1, 2, \ldots, n$. We first show that when k and n are both even, Player B has a winning strategy. In this case we can divide the squares into disjoint adjacent pairs $\{2i-1, 2i\}$ with $1 \leq i \leq \frac{n}{2}$. At the beginning of the game the pawns completely occupy the squares in the leftmost $k/2$ pairs, and all the squares in the remaining pairs are vacant. Thus Player A's first move must take a pawn from an occupied pair of squares and place it in one of a vacant pair of squares. A winning strategy for player B is to always take the other pawn of the pair that Player A moved from and place it in the remaining square of the pair that Player A moved to. Thus after each of Player B's moves, each of the pairs Pi either has pawns in both squares or in neither, whereas after each of Player A's moves, there are two of the pairs Pi with one pawn each. In particular, Player A can never reach the ending position, and the game will end after one of Player B's moves. If k and n are not both even, Player A always has a first move available which will leave Player B either with no moves at all, or with a position equivalent to the starting position of our game with even integers k_1 and n_1, $1 \leq k_1 < n_1$. Thus by the case discussed in the previous paragraph, Player A (as the second player from that position) has a winning strategy. Specifically, if k and n are both odd, Player A can move the pawn in square k to square n, leaving $k_1 = k - 1$ pawns at the beginning of a line of $n_1 = n - 1$ remaining squares. (If $k = 1$, the game is then over.) If k is odd and n is even, Player A can move the pawn in square 1 to square n, winning the game immediately if $k = 1$ and otherwise leaving $k_1 = k - 1$ pawns at the beginning of a line of $n_1 = n - 2$ remaining squares. Finally, if k is even and n is odd, Player A can move the pawn in square 1 to square $k + 1$, winning the game immediately if $n = k + 1$ and otherwise leaving $k_1 = k$ pawns at the beginning of a line of $n_1 = n - 1$ remaining squares. In each of these three cases, after making the indicated first move, Player A can use Player B's strategy from the previous paragraph to win.