**MATERIAŁY NA XXXVII KONFERENCJĘ**
**Z GEOMETRII ANALITYCZNEJ I ALGEBRAICZNEJ**

**2016** **Łódź** **str. 9**

# ISOLATED HOMOGENEOUS AND SEMI-HOMOGENEOUS HYPERSURFACE SINGULARITIES

Szymon Brzostowski and Tadeusz Krasiński[1] (Łódź)

**Abstract**

We show how Zuo and Yau's characterizations of homogeneous and semi-homogeneous hypersurface singularities easily follow from the standard theorems of multidimensional complex analysis.

## 1 Introduction

Homogeneous and weighted homogeneous (isolated) singularities play an important role in singularity theory (see [Bri66], [Pha65]). Accordingly, the question of finding possible characterizations of various „kinds" of these singularities is both an important and interesting problem from the point of view of singularity theory. An interesting characterization of weighted homogeneous singularities was given by Saito in 1971 (see Section 3). Characterizations of homogeneous and semi-homogeneous singularities in terms of some invariants were given in a series

[1]This research was partially supported by OPUS Grant No 2012/07/B/ST1/03293 (Poland).

of papers by Xu, Yau, Lin and Zuo, the first of which appeared already in 1993. The final general result was proved by Yau and Zuo in 2015 [YZ15] (details are in Section 3). Another proof of their result was recently presented by Abderrahmane [Abd15]. The aim of this paper is to indicate that these characterizations are immediate corollaries from a well-known theorem of multidimensional complex analysis (more specifically from multiplicity theory of mappings) given by Zariski, Tsikh and Yuzhakov. Section 2 is devoted to clarification of the notion of weighted homogeneous polynomials (known also as quasihomogeneous) as there are various definitions of this concept. In Section 3 we describe the history of known characterizations of homogeneous and semi-homogeneous isolated singularities. In Section 4 we cite the aforementioned theorem on multiplicity of finite holomorphic maps between spaces of the same dimensions. In Section 5 we obtain characterizations of homogeneous and semi-homogeneous isolated singularities (given by Yau and Zuo) as simple consequences of the result of Section 4. Moreover, we obtain a stronger version of this characterization.

## 2 Weighted homogeneous polynomials

In this section we collect basic information on weighted homogeneous (or quasi-homogeneous) polynomials. Since in literature there are various definitions of these, we fix the following.

**Definition 1** *A polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ is weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (weights) such that for each monomial $z_1^{i_1} \cdot \ldots \cdot z_n^{i_n}$ appearing in $P$ with a non-zero coefficient it holds*

$$w_1 \cdot i_1 + \ldots + w_n \cdot i_n = 1.$$

Notice that if for a variable $z_k$ we have $i_k \neq 0$ (so that $P$ effectively depends on $z_k$), then $w_k \leqslant 1$. *Therefore, we will always assume that* $w_k \leqslant 1$ $(k = 1, \ldots, n)$.

**Remark** We work over $\mathbb{C}$ here, because we only want to apply the results of this section to complex singularities. For the most part, there is no essential difference in the treatment if one replaces $\mathbb{C}$ with a field $\mathbb{K}$.

If a weight satisfies the inequality

$$0 < w_i \leqslant \tfrac{1}{2},$$

we call it a strong weight. Otherwise, if

$$\tfrac{1}{2} < w_i \leqslant 1,$$

then this weight $w_i$ is called weak. If at least one weight is weak we say that the weights of $P$ are weak.

If $P$ is homogeneous, then $P$ is weighted homogeneous with weights $w_1 = \ldots = w_n = \frac{1}{\deg P}$. More generally, any polynomial (or a power series even) can be represented as a sum of weighted homogeneous polynomials. In order to be able to refer to this representation in a natural way, it is desirable to introduce the notion of a *weighted degree*. This leads to the following, more flexible definition of the concept of weighted homogeneous polynomials.

**Definition 2** *A polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ is weighted homogeneous of degree $d$ if there exist positive integers $\omega_1, \ldots, \omega_n$ (weights) and $d$ (degree) such that $\gcd(\omega_1, \ldots, \omega_n) = 1$ and for each monomial $z_1^{i_1} \cdot \ldots \cdot z_n^{i_n}$ appearing in $P$ with a non-zero coefficient it holds*

$$\omega_1 \cdot i_1 + \ldots + \omega_n \cdot i_n = d.$$

*The integer $d$ is called the weighted degree of $P$ with respect to the system of weights $\omega = (\omega_1, \ldots, \omega_n)$ and is denoted by $\deg_\omega P$.*

It is not hard to see that both definitions introduce one and the same concept, and can be used interchangeably, depending on one's needs. Namely, if $P$ is weighted homogeneous of degree $d$ with weights $\omega_1, \ldots, \omega_n$ (in the sense of Definition 2), then $P$ is also weighted homogeneous with weights $w_i := \frac{\omega_i}{d}$ in the sense of Definition 1. On the other hand, if $w_1, \ldots, w_n$ are the weights of $P$ as in Definition 1 and we write $w_i = \frac{a_i}{b_i}$, where $a_i$ and $b_i$ are co-prime positive integers, then upon putting $d := \operatorname{lcm}(b_1, \ldots, b_n)$ and $\omega_i := \frac{a_i}{b_i} d$ $(i = 1, \ldots, n)$ we can interpret $P$ as being weighted homogeneous of degree $d$, in the sense of the second definition. Note also that in this correspondence we have $w_i \leqslant 1/2 \Leftrightarrow \omega_i \leqslant d/2$ and $w_i \leqslant 1 \Leftrightarrow \omega_i \leqslant d$. Consequently, to be consistent with the agreement that we made before, we will also *always assume that $\omega_i \leqslant d$.*

Equipped with this improved language, we may now easily express any formal power series $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ as a formal sum of weighted homogeneous polynomials of distinct increasing degrees,

$$f = f_{d_1} + f_{d_2} + \ldots, \quad d_1 < d_2 < \ldots;$$

namely we just collect the terms of same weighted $\omega$-degree into polynomials $f_d$. This is always possible, for: (1) only finitely many $n$-tuples of non-negative integers $(i_1, \ldots, i_n)$ satisfy the relation $\omega_1 \cdot i_1 + \ldots + \omega_n \cdot i_n = d$, (2) these tuples have the property that $\|(i_1, \ldots, i_n)\|_\infty \to \infty$ with $d \to \infty$. Some of these $f_d$'s may be zero and then we can safely exclude them from the representation of $f$, but in order to make such a representation truly unique we would have to assume that $f_{d_j} \neq 0$ for all $j \geqslant 0$, which is sometimes inconvenient. Anyways, if $f_{d_1} \neq 0$ then we call this polynomial the initial part of $f$ with respect to the weights $\omega_1, \ldots, \omega_n$ and we denote it $\operatorname{in}_\omega f$. We can also use the notation $\operatorname{in}_w f$, where $w_i = \frac{\omega_i}{d}$ are the corresponding rational weights. In particular, by this definition, we always have $\operatorname{in}_w f = \operatorname{in}_\omega f \neq 0$ if $f \neq 0$.

There is a clear geometric picture connected with weighted homogeneous polynomials. Namely, let $w_1, \ldots, w_n$ be the rational weights of $P$ and let

$$\operatorname{supp} P := \left\{(i_1, \ldots, i_n) \in \mathbb{N}_0^n : z_1^{i_1} \cdot \ldots \cdot z_n^{i_n} \text{ appears in } P \text{ with a non-zero coeff.}\right\}$$

be the support of $P$. Then all the points of $\operatorname{supp} P$ lie in the hyperplane in $\mathbb{R}^n$ defined by the equation

$$w_1 \cdot x_1 + \ldots + w_n \cdot x_n = 1,$$

and this hyperplane intersects the respective coordinate axes exactly at distances $1/w_1, \ldots, 1/w_n$ from the origin.

Now, let us consider the problem of uniqueness of the (rational) weights of a given weighted homogeneous polynomial $P$. Simple examples show that in general $P$ does not determine its weights. For instance, let $P(z_1, z_2) := z_1 \cdot z_2$. There are many possible choices for weights of $P$, e.g. $w_1 = w_2 = \frac{1}{2}$ or $w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$. However, if we assume that $P$ defines an isolated singularity at the origin, then one can prove a simple criterion for the uniqueness of the weights. To this end, let us first recall the notion of an isolated singularity.

A holomorphic function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ (the local ring of all such germs at 0 will be denoted by $\mathcal{O}_n$) is called an isolated singularity (or defines an isolated singularity) if $f$ has an isolated critical point at 0 i.e.

1. $\nabla f := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ vanishes at 0,
2. $\nabla f(\mathbb{z}) \neq 0$ for small $\mathbb{z} \in \mathbb{C}^n$, $\mathbb{z} \neq 0$.

If $f(\mathbb{z}) := \sum_{\mathbb{i} \in \mathbb{N}_0^n} a_{\mathbb{i}} \mathbb{z}^{\mathbb{i}}$, where $\mathbb{i} = (i_1, \ldots, i_n)$, $\mathbb{z} = (z_1, \ldots, z_n)$ and $\mathbb{z}^{\mathbb{i}} := z_1^{i_1} \cdot \ldots \cdot z_n^{i_n}$, then its support $\operatorname{supp} f = \{\mathbb{i} \in \mathbb{N}_0^n : a_{\mathbb{i}} \neq 0\}$ has a simple feature.

**Lemma 1** *If $f \in \mathcal{O}_n$ is an isolated singularity, then for each $k = 1, \ldots, n$ a monomial of the form $z_k^{i_k}$, where $i_k \geqslant 2$, or $z_k^{i_k} z_j$, where $i_k \geqslant 1$, $k \neq j$, appears with a non-zero coefficient in the power series expansion of $f$.*

**Proof** Immediate, for otherwise all the partial derivatives $\frac{\partial f}{\partial z_i}$ would vanish along the axis $Oz_k$. □

**Proposition 1** *If a weighted homogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ defines an isolated singularity at 0 and $\operatorname{ord} P \geqslant 3$, then its weights are unique and strong. Moreover, $w_i < \frac{1}{2}$ $(i = 1, \ldots, n)$.*

**Proof** Consider one of the weights; let this be $w_1$ for simplicity. Because of Lemma 1 and since $\operatorname{ord} P \geqslant 3$, in the expansion of $P$ there appears a monomial of the form $z_1^{i_1}$, where $i_1 \geqslant 3$, or $z_1^{i_1} z_j$, where $i_1 \geqslant 2$, $1 \neq j$. In the first case we get $w_1 = \frac{1}{i_1} < \frac{1}{2}$, so then this weight is uniquely determined by $P$ itself. In the opposite case, upon rearranging the variables, we may assume that $z_1^{i_1} z_2$ appears in $P$ with a non-zero coefficient. Repeating the above reasoning for the variable $z_2$ we infer that either $w_2 = \frac{1}{i_2} < \frac{1}{2}$ for some $i_2 \geqslant 3$, and then we also get $w_1 = \frac{1-w_2}{i_1} < \frac{1}{2}$ so that the weight $w_1$ is uniquely determined, or in $P$ there appears a monomial $z_2^{i_2} z_j$, where $i_2 \geqslant 2$, $2 \neq j$, with a non-zero coefficient. In this second case it may happen that $j \neq 1$. Then, again, we may assume that $j = 3$. Continuing this procedure, after a finite number of steps either we will find that actually $w_1$ is uniquely determined by $P$ or we will find „a cycle" of monomials appearing in $P$ with non-zero coefficients:

$$z_1^{i_1} z_2, z_2^{i_2} z_3, \ldots, z_k^{i_k} z_j, \text{ where } i_1, \ldots, i_k \geqslant 2, j \in \{1, \ldots, k-1\}, k \geqslant 2.$$

This leads to the following linear system of equations in $w_1, \ldots, w_k$:

$$\begin{cases} i_1 w_1 + w_2 & = & 1 \\ i_2 w_2 + w_3 & = & 1 \\ \quad \vdots & & \vdots \\ i_k w_k + w_j & = & 1 \end{cases} .$$

It is easy to check that the determinant of this system is equal to $i_1 \ldots i_{j-1}(i_j \ldots i_k + (-1)^{k+j})$, so it is non-zero because $i_k \geqslant 2$. Hence, the weights $w_1, \ldots, w_k$ are uniquely determined. Moreover, using the first equation, we see that $w_1 = \frac{1-w_2}{i_1} < \frac{1}{2}$. Since $w_1$ was an arbitrarily fixed weight of $P$, the result follows. □

**Remark** Note that in the proof of Proposition 1 we actually showed that the weights $w_i$ have to be rational, even if one allowed $w_i$ to be real in Definition 1 (subject to the condition $0 < w_i \leqslant 1$). The same remark concerns Proposition 2 below.

**Example 1** The above result does not hold for isolated singularities of order 2.

- Take $P(z_1, z_2) := z_1 z_2$. As we saw on page 12, the weights of $P$ are not uniquely determined.

- Take $Q(z_1, z_2) := z_1 z_2 + z_2^3$. Then the weights of $Q$ are indeed unique, but not strong.

A simple analysis of the proof of Proposition 1 leads to the following observation valid also for order 2 weighted homogeneous polynomials defining isolated singularities.

**Proposition 2 ([Sai71, Korollar 1.7])** *If a weighted homogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ with strong weights defines an isolated singularity at 0, then there are no other strong weights for $P$.*

**Proof** We repeat the proof of Proposition 1 to get the „cycle" $z_1^{i_1}z_2, z_2^{i_2}z_3, \ldots, z_k^{i_k}z_j$ of monomials appearing with non-zero coefficients in $P$ and the corresponding linear system

$$\begin{cases} i_1w_1 + w_2 &= 1 \\ i_2w_2 + w_3 &= 1 \\ \quad\vdots & \ \vdots \\ i_kw_k + w_j &= 1 \end{cases},$$

where this time we know only that $i_1, \ldots, i_k \geqslant 1, j \in \{1, \ldots, k-1\}, k \geqslant 2$. Hence, its determinant $i_1 \ldots i_{j-1}(i_j \ldots i_k + (-1)^{k+j})$ may be zero if $k + j \equiv 1 (\mathrm{mod}\, 2)$ and $i_j = \ldots = i_k = 1$. But if this is the case, then in $P$ there appear the monomials $z_jz_{j+1}, \ldots, z_{k-1}z_k, z_kz_j$ with non-zero coefficients, implying $w_j = \ldots = w_k = \frac{1}{2}$ as the weights we consider are strong. Thus, we are left with the system $\left\{i_1w_1 + w_2 = \ldots = i_{j-1}w_{j-1} + w_j = 1, w_j = \frac{1}{2}\right\}$, which is a triangular one. Hence, the uniqueness of $w_1, \ldots, w_n$ follows. □

Without the assumption that the weights are strong, the following holds „in order 2".

**Proposition 3 (cf. [Sai71, the proof of Satz 1.3])** *If a weighted homogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ defines an isolated singularity at* 0 *and* $\mathrm{ord}\, P = 2$, *then there exists a biholomorphic change of coordinates $\Phi$ of $\mathbb{C}^n$ such that $P \circ \Phi$ is still weighted homogeneous, but with unique and strong weights.*

**Proof** We apply induction with respect to the number $n$ of variables. For $n = 1$ the assertion is obvious. Let now $P \in \mathbb{C}[z_1, \ldots, z_n]$, $n \geqslant 2$, define a weighted homogeneous singularity. Let $w_1, \ldots, w_n$ be the weights of $P$. Assume that in $P$ there appears a monomial $z_jz_k$, for some $j, k \in \{1, \ldots, n\}$, with a non-zero coefficient. If $j = k$ then clearly $w_j = \frac{1}{2}$; if $j \neq k$ with $w_j \leqslant \frac{1}{2}$ and $w_k \leqslant \frac{1}{2}$, then we also immediately conclude that $w_j = w_k = \frac{1}{2}$. Hence, in these two cases the weights $w_j, w_k$ are already strong. Accordingly, assume now that $j \neq k$ and e.g. $w_j > \frac{1}{2}$. By a permutation of the variables, we may arrange things so that $j = 1$, $k = 2$. Since $w_1 > \frac{1}{2}$, $P$ depends only linearly on $z_1$ and we may write

$$P(z_1, \ldots, z_n) = z_1(az_2 + P_0(z_3, \ldots, z_n)) + P'(z_2, \ldots, z_n), \quad a \neq 0.$$

Here, $P_0$ also does not depend on $z_2$ because $P$ is weighted homogeneous; moreover $\mathrm{ord}\, P_0 \geqslant 1$. If we change the coordinates in the following manner: $t_1 = z_1, t_2 = az_2 + P_0(z_3, \ldots, z_n), t_3 = z_3, \ldots, t_n = z_n$, then we obtain a new weighted homogeneous polynomial $Q$ having the same weights $w_1, \ldots, w_n$ and of the form

$$Q(t_1, \ldots, t_n) = t_1t_2 + Q'(t_2, \ldots, t_n).$$

Now, we have the representation

$$Q(t_1, \ldots, t_n) = t_2(t_1 + Q_0(t_2, \ldots, t_n)) + Q''(t_3, \ldots, t_n)$$

and the substitution $u_1 = t_1 + Q_0(t_2, \ldots, t_n), u_2 = t_2, \ldots, u_n = t_n$ leads to another weighted homogeneous polynomial $R$ having the same weights $w_1, \ldots, w_n$ and of the form

$$R(u_1, \ldots, u_n) = u_1 u_2 + R'(u_3, \ldots, u_n).$$

But now we can put $w_1' := \frac{1}{2}$, $w_2' := \frac{1}{2}$, and then both $R$ and $R'$ are weighted homogeneous with respect to $w_1', w_2', w_3, \ldots, w_n$. Moreover, $R'$ defines an isolated singularity in $\mathbb{C}^{n-2}$. If we change coordinates: $u_1 = s_1 - is_2$, $u_2 = s_1 + is_2$, $u_3 = s_3, \ldots, u_n = s_n$ we obtain a weighted homogeneous polynomial

$$T(s_1, \ldots, s_n) = s_1^2 + s_2^2 + R'(s_3, \ldots, s_n)$$

for which the weights of the variables $s_1$ and $s_2$ are unique and strong. Induction finishes the proof. □

**Remark** A careful analysis of the above proof shows that, in order to guarantee *the existence* of strong weights for $P$, one must only change these variables $z_{i_1}, \ldots, z_{i_r}$ which appear in the quadratic form of $P$ exclusively as $z_{i_j} z_{i_k}$, with $\max(w_{i_j}, w_{i_k}) > \frac{1}{2}$. However, to be able to deduce *the uniqueness* of the weights, one must work with all the variables appearing in the quadratic form of $P$. In both cases, all the other variables remain essentially uninfluenced by this procedure. Moreover, $w_\alpha$ ($\alpha \notin \{i_1, \ldots, i_r\}$) are the same in all these coordinate systems.

We may sum up the above propositions in one theorem.

**Theorem 1** *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a weighted homogeneous polynomial defining an isolated singularity at 0.*

1. *If $\operatorname{ord} P \geqslant 3$, then the weights of $P$ are unique and strong (and, moreover, all $w_i < \frac{1}{2}$).*
2. *If $\operatorname{ord} P = 2$, then there exists a biholomorphic change of coordinates of $\mathbb{C}^n$ such that $P$ in these coordinates is weighted homogeneous with unique and strong weights (and might be brought to the form $P(s_1, \ldots, s_n) = s_1^2 + \ldots + s_k^2 + P'(s_{k+1}, \ldots, s_n)$, $k \geqslant 0$, where $P'$ is weighted homogeneous and $\operatorname{ord} P' \geqslant 3$).*
3. *If $P$ has strong weights, then they are the only strong weights for $P$.*

**Remark** Analyzing the proofs given, a careful reader might notice that Theorem 1 holds also for a weighted homogeneous polynomial with a non-isolated critical point at 0, as long as this polynomial satisfies the assertion of Lemma 1 (that is, if it is *nearly convenient*).

**Example 2** As we saw in Example 1, the polynomial $Q(z_1, z_2) := z_1 z_2 + z_2^3$ has unique, but weak, weights. Putting $u_1 = z_1 + z_2^2$, $u_2 = z_2$ we transform $Q$ into the form $u_1 u_2$. Another change: $s_1 = u_1 + u_2$, $s_2 = u_1 - u_2$ leads to $\frac{1}{4}s_1^2 - \frac{1}{4}s_2^2$, which is now a weighted homogeneous polynomial with unique and strong weights.

# 3 Characterization of homogeneous and weighted homogeneous isolated singularities

In this section we describe some known characterizations of homogeneous and semi-homogeneous isolated singularities. To make things precise, let us start the discussion with necessary definitions.

**Definition 3** *A function-germ $f \in \mathcal{O}_n$ is called a homogeneous (resp. weighted homogeneous) isolated singularity if $f$ is a homogeneous (resp. weighted homogeneous) polynomial defining an isolated singularity at $0 \in \mathbb{C}^n$. Such germ $f$ is called a semi-homogeneous (resp. semi-weighted-homogeneous) singularity if $f = f_0 + f'$, where $f_0$ (the principal part of $f$) is a homogeneous (resp. weighted homogeneous) isolated singularity and every monomial appearing in $f'$ with a non-zero coefficient has its degree (resp. weighted degree) greater than the degree (resp. weighted degree) of $f_0$.*

The most impressive description of the class of weighted homogeneous isolated singularities was given by K. Saito in 1971.

**Theorem 2 ([Sai71, Satz 4.1])** *An isolated singularity $f \in \mathcal{O}_n$ is weighted homogeneous in some system of coordinates if, and only if, $f \in (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})\mathcal{O}_n$.*

We remark that the „only if" part of this theorem is easy and follows directly from Euler's formula for weighted homogeneous polynomials.

In singularity theory there are many numerical invariants in terms of which various properties of isolated singularities (e.g. of topological, geometric or holomorphic nature) can be expressed. The most important of these are perhaps: the multiplicity $m(f)$, the Milnor number $\mu(f)$ and the Tjurina number $\tau(f)$. They can be defined as follows:

$$\begin{aligned} m(f) &= \operatorname{ord}(f), \\ \mu(f) &= \dim_{\mathbb{C}} \mathcal{O}_n/(\tfrac{\partial f}{\partial z_1}, \ldots, \tfrac{\partial f}{\partial z_n})\mathcal{O}_n, \\ \tau(f) &= \dim_{\mathbb{C}} \mathcal{O}_n/(f, \tfrac{\partial f}{\partial z_1}, \ldots, \tfrac{\partial f}{\partial z_n})\mathcal{O}_n. \end{aligned}$$

Using these invariants, we may reformulate Saito theorem.

**Theorem 2'** *An isolated singularity $f \in \mathcal{O}_n$ is weighted homogeneous in some system of coordinates if, and only if, $\mu(f) = \tau(f)$.*

This result justifies the hope for, and inspires the search for, a possible characterization of homogeneous and semi-homogeneous isolated singularities, one that should also be expressed only in terms of the above-mentioned numerical invariants. In recent years a number of papers concerning this topic have appeared. Xu and

Yau in 1993 [XY93] managed to solve this problem for 2-dimensional isolated singularities, after that Lin and Yau [LY04], and Chen et al. [CLYZ11] extended the first result to 3- and 4-dimensional isolated singularities, respectively. Since these characterizations were similar, in 2006 Yau formulated the following conjecture (see [LWYL06]):

**Conjecture** *Let $f \in \mathcal{O}_n$ be an arbitrary isolated singularity. Then*

1. *$\mu(f) \geqslant (m(f)-1)^n$, with equality if, and only if, $f$ is semi-homogeneous.*

*Assume additionally that $f$ is a weighted homogeneous isolated singularity. Then*

2. *$\mu(f) = (m(f)-1)^n$ if, and only if, there is a biholomorphic change of coordinates in $\mathbb{C}^n$ which transforms $f$ into a homogeneous polynomial (so that $f$ becomes a homogeneous isolated singularity in these new coordinates).*

The first part of this general conjecture was proved by Yau and Zuo in [YZ12], and the second part by the same authors in [YZ15]. Quite recently, Abderrahmane [Abd15] gave another proof of this conjecture.

The aim of this elaboration is to indicate that the conjecture is a simple corollary to a well-known theorem of multidimensional complex analysis (more specifically – multiplicity theory of mappings). Moreover, we will give a more precise version of the second part of the conjecture. In a similar way one can prove the result of Furuya and Tomari [FT04] on the characterization of semi-weighted-homogeneous isolated singularities.

Let us prepare the ground first.

# 4 Elements of multiplicity theory

Let $F = (F_1, \ldots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map having an isolated zero at $0 \in \mathbb{C}^n$. In such a case $F$ is a branched analytic cover, so for all $y$ in a dense open subset of a suitable neighbourhood of 0 the number of points in $F^{-1}(y)$ is finite and independent of $y$. Hence, one may define the covering multiplicity of $F$ at 0 by

$$\mu_c(F) := \#\left\{F^{-1}(y) : y - \text{generic}\right\}.$$

A standard result in complex analytic geometry (see [Pal67], [Orl77, Thm. I.5.13], [Tsi92, p. 148]) states that

$$\mu_c(F) = \dim_{\mathbb{C}} \mathcal{O}_n/(F_1, \ldots, F_n)\mathcal{O}_n.$$

Hence, in the case where $F = \nabla f$ for an isolated singularity $f$, this gives (cf. page 16)

$$\mu(f) = \mu_c(\nabla f).$$

Another well-known result on multiplicity is the following, dating back at least to O. Zariski [Zar37], who proved it for polynomials (not a restrictive assumption) using properties of multipolynomial resultants.

**Theorem 3 ([TY78], [Tsi92, p. 146], [AY83, p. 181], [Chi89, p. 112], [ATY94, p. 37])** *Let $F = (F_1, \ldots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map with an isolated zero at $0 \in \mathbb{C}^n$. Let $\operatorname{in} F = (\operatorname{in} F_1, \ldots, \operatorname{in} F_n)$ be the vector of the initial forms of the mapping $F$, of degrees $d_1, \ldots, d_n$, respectively. Then*

1. $\mu_c(F) \geqslant d_1 \cdot \ldots \cdot d_n$.
2. *The equality in 1. holds if, and only if,* $0$ *is an isolated zero of the system* $\operatorname{in} F$.

Theorem 3 admits a simple generalization.

**Theorem 4 ([TY78], [AY83, p. 184], [Chi89, p. 114])** *Let $F = (F_1, \ldots, F_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map with an isolated zero at $0 \in \mathbb{C}^n$. Let $\omega_1, \ldots, \omega_n$ be some fixed natural weights. Let $\operatorname{in}_\omega F = (\operatorname{in}_\omega F_1, \ldots, \operatorname{in}_\omega F_n)$ be the vector of the weighted initial forms of the mapping $F$, of weighted degrees $d_1, \ldots, d_n$, respectively. Then*

1. $\mu_c(F) \geqslant d_1 \cdot \ldots \cdot d_n$.
2. *The equality in 1. holds if, and only if,* $0$ *is an isolated zero of the system* $\operatorname{in}_\omega F$.

Let us remark that Theorem 4 follows from Theorem 3 by means of the simple substitution $\Phi : \mathbb{z} \mapsto (z_1^{\omega_1}, \ldots, z_n^{\omega_n})$, which transforms the weighted $\operatorname{in}_\omega F$ into the homogeneous $\operatorname{in} \Phi^* F$.

# 5 Corollaries

Let us relate the theorems from the previous section to the Conjecture.

I. Conjecture, item 1, follows immediately from Theorem 3.

II. Conjecture, item 2 also follows from Theorem 3, and in a more precise form, as we are about to see.

**Theorem 5** *Let $f \in \mathcal{O}_n$ be a weighted homogeneous isolated singularity. Then $\mu(f) = (m(f) - 1)^n$ if, and only if, there is a biholomorphic change of coordinates in $\mathbb{C}^n$ which transforms $f$ into a homogeneous polynomial. Moreover, such change of coordinates may be needed only when* $\operatorname{ord} f = 2$ *and all possible systems of weights for $f$ are weak.*

**Proof** If the equality holds, then, according to Theorem 3, $f$ is semi-homogeneous. Let $m = m(f)$, then $f = f_m + f_{m+1} + \dots$ where each $f_i$ is homogeneous of degree $i$, and $f_m$ is an isolated singularity. Since $f$ is weighted homogeneous, $f_m$ also is, with respect to the same weights. Now, if the weights of $f$ are strong, then from Proposition 2 applied to $f_m$ we infer that these weights must be $\frac{1}{m}, \dots, \frac{1}{m}$. Hence, $f_{m+1} = f_{m+2} = \dots = 0$ and $f = f_m$ is homogeneous. If the weights of $f$ are weak, then Proposition 1 implies that $\operatorname{ord} f = 2$, and Proposition 3 supplies us with a biholomorphic change of coordinates which makes $f$ weighted homogeneous, with strong weights. Since both, the Milnor number, and the multiplicity are invariants of biholomorphisms, the equality still holds in this new system of coordinates. Hence, we may apply the first part of the proof to conclude that $f$ is now homogeneous. □

III. In 2004, in *Proc. AMS*, Furuya and Tomari [FT04] proved the following theorem, which is a direct consequence of Theorem 4.

**Theorem 6** *Let $f \in \mathcal{O}_n$ be an isolated singularity and let $\omega_1, \dots, \omega_n$ be some fixed natural weights. With respect to these weights, write $f = f_d + f_{d+1} + \dots$, where $f_i$ is a weighted homogeneous polynomial of weighted degree equal to $i$, and $f_d \neq 0$. Then*

1. $\mu(f) \geqslant \left(\frac{d}{\omega_1} - 1\right) \dots \left(\frac{d}{\omega_n} - 1\right)$.
2. *The equality holds in the above if, and only if, $f$ is semi-weighted-homogeneous with respect to the weights $\omega$.*

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Szymon Brzostowski
Faculty of Mathematics
and Computer Science
University of Łódź
ul. Banacha 22,
90-238 Łódź, Poland
brzosts@math.uni.lodz.pl

Tadeusz Krasiński
Faculty of Mathematics
and Computer Science
University of Łódź
ul. Banacha 22,
90-238 Łódź, Poland
krasinsk@uni.lodz.pl

OSOBLIWOŚCI IZOLOWANE HIPERPOWIERZCHNI JEDNORODNYCH I SEMI-JEDNORODNYCH

**Streszczenie** Wskazujemy w jaki sposób twierdzenia charakteryzacyjne dla osobliwości izolowanych hiperpowierzchni jednorodnych i semi-jednorodnych, podane ostatnio przez Yau i Zuo, wywnioskować można niemal natychmiast ze standardowych twierdzeń wielowymiarowej analizy zespolonej.

*Łódź, 11 – 15 stycznia 2016 r.*