MATERIAŁY NA XXXIII KONFERENCJĘ I WARSZTATY
Z GEOMETRII ANALITYCZNEJ I ALGEBRAICZNEJ

2012 **Łódź** **str. 9**

THE JUMP OF THE MILNOR NUMBERS IN THE X_9 SINGULARITY CLASS

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Abstract

The jump of Milnor numbers of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations (f_s). We prove that for the singularities $x^4 + y^4 + ax^2y^2$, where $a \in \mathbb{C}, a^2 \neq 4$, of the X_9 singularity class the jump of Milnor numbers is equal to 2.

1 Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *(isolated) singularity*, i.e. f_0 is a germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \to \mathbb{C}$ of f_0, holomorphic in an open neighborhood U of the point $0 \in \mathbb{C}^n$, such that:

1. $\hat{f}_0\,(0) = 0$,
2. $\nabla \hat{f}_0\,(0) = 0$,
3. $\nabla \hat{f}_0\,(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$.

In the sequel we will identify germs of holomorphic functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of n variables will be denoted by $\mathcal{O}^n$.

A *deformation of the singularity* f_0 is the germ of a holomorphic function $f = f(s,z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that:

1. $f(0,z) = f_0(z)$,
2. $f(s,0) = 0$,
3. for each $|s| \ll 1$ it is $\nabla_z f(s,z) \neq 0$ for $z \neq 0$ in a (small) neighborhood of $0 \in \mathbb{C}^n$.

The deformation $f(s,z)$ of the singularity f_0 will also be treated as a family (f_s) of germs, taking $f_s(z) := f(s,z)$. In this context, the symbol ∇f_s will always denote $\nabla_z f_s(\mathrm{z})$.

Remark. Notice that in the deformation (f_s) there can occur in particular *smooth* germs, that is germs satisfying $\nabla f_s(0) \neq 0$.

By the above assumptions it follows that, for every sufficiently small s, one can define a (finite) number μ_s as the Milnor number of f_s, namely

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}^n / (\nabla f_s) = i_0 \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right),$$

where the symbol $i_0 \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$ denotes the multiplicity of the ideal $\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}^n$.

Since the Milnor number is upper semi-continuous in families of singularities [GLS07, Ch. I, Thm. 2.6], there exists an open neighborhood S of the point $0 \in \mathbb{C}$ such that

1. $\mu_s = \mathrm{const.}$ for $s \in S \setminus \{0\}$,
2. $\mu_0 \geqslant \mu_s$ for $s \in S$.

The (constant) difference $\mu_0 - \mu_s$ for $s \in S \setminus \{0\}$ will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest nonzero value among all the jumps of deformations of the singularity f_0 will be called *the jump of the singularity* f_0 and denoted by $\lambda(f_0)$.

The first general result concerning the problem of computation of the jump was given by S. Gusein-Zade [Gus93], who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$. He showed that a generic element in some classes of singularities (satisfying conditions concerning the Milnor numbers and modality) fulfills $\lambda(f_0) > 1$, but he didn't give any particular example of such a singularity.

The two-dimensional version of the problem of computation of the jump, and more precisely – of *the non-degenerate jump* (i.e. all the families (f_s) being considered are to be made of Kouchnirenko non-degenerate singularities), has been studied in the following papers: [Bod07], [Wal08], [Wal09], [Wal10], [Wal12].

The following are example singularities that fulfill the assumptions of the Gusein-Zade theorem.

1. $x^4 + y^4$ – a singularity of modality 1. Corresponding to it is the class of singularities with constant Milnor number and of modality 1, namely
$$x^4 + y^4 + ax^2y^2, \quad a^2 \neq 4, \quad \mu_a = 9.$$
It is the class X_9 in the terminology of [AGV85].

2. $x^4 + y^6$ – a singularity of modality 2. Corresponding to it is the class of singularities with constant Milnor number and of modality 2, namely
$$x^4 + y^6 + (a + by)\, x^2y^3, \quad a^2 \neq 4, \quad \mu_{ab} = 15.$$
It is the class $W_{1,0}$ in the terminology of [AGV85].

3. $x^3 + y^9$ – a singularity of modality 2. Corresponding to it is the class of singularities with constant Milnor number and of modality 2, namely
$$x^3 + y^9 + ax^2y^3 + bxy^7, \quad 4a^3 + 27 \neq 0, \quad \mu_{ab} = 16.$$
It is the class $J_{3,0}$ in the terminology of [AGV85].

What one can conclude is that generic elements f of the classes X_9, $W_{1,0}$, $J_{3,0}$ mentioned above satisfy $\lambda(f) > 1$. However, determining the jump of any particular element of these classes is still an open problem and in fact Gusein-Zade did not give any specific example of a singularity f with $\lambda(f) > 1$. The purpose of this work is to prove that for the singularities f_0 in the X_9 class
$$f_0(x, y) = x^4 + y^4 + ax^2y^2, \quad a \in \mathbb{C}, a^2 \neq 4,$$
it is
$$\lambda(f_0) = 2$$
(and that therefore all the singularities of the class X_9 are "generic" in the family X_9) and for the following singularities in the $W_{1,0}$ class
$$f_0(x, y) = x^4 + y^6 + bx^2y^4, \quad b \in \mathbb{C}$$
it is
$$\lambda(f_0) = 1$$
(and that therefore these singularities are not "generic" in the family $W_{1,0}$).

We also pose some open problems:

1. Show that for the remaining singularities in the $W_{1,0}$ class, i.e. for the singularities $f^{(a,b)} := x^4 + y^6 + (a + by)\, x^2y^3$, where $a, b \in \mathbb{C}, 0 \neq a^2 \neq 4$, it is $\lambda\left(f^{(a,b)}\right) = 2$.

2. Compute the jumps for the singularities $f^{(a,b)}$ in the class $J_{3,0}$ with respect to the parameters a, b.

2 Introductory Facts

In this section we review briefly the notion of non-degeneration of singularity and the known theorems of Kouchnirenko and Płoski on the Milnor numbers of non-degenerate singularities, as well as Bodin's results about the non-degenerate jumps of singularities. Here we restrict ourselves to considering the two-dimensional case, as that is what will be needed in the sequel. However, at the end of the section there is also discussed the notion of a versal unfolding, and the fundamental theorem on it is given, and we work in n–dimensions in this context.

In the following we define $\mathbb{N}$ to be the set of nonnegative integers, and $\mathbb{R}_+$ will denote the set of nonnegative real numbers. Let $f_0(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij}x^iy^j$ be a singularity. Let $\operatorname{supp}(f_0) := \left\{(i,j) \in \mathbb{N}^2 : a_{ij} \neq 0\right\}$. The *Newton Diagram of* f_0 is defined as the convex hull of the set

$$\bigcup_{(i,j)\in\operatorname{supp}(f_0)} (i,j) + \mathbb{R}^2_+$$

and is denoted by $\Gamma_+(f_0)$. It is easy to see that the boundary (in $\mathbb{R}^2$) of the diagram $\Gamma_+(f_0)$ is a sum of two half-lines and a finite number of compact line segments (a degenerate case of no segments included). The set of those line segments will be called a *Newton Polygon of the singularity* f_0 and denoted by $\Gamma(f_0)$. For each segment $\gamma \in \Gamma(f_0)$ we define a weighted homogenous polynomial

$$(f_0)_\gamma := \sum_{(i,j)\in\gamma} a_{ij}x^iy^j.$$

A singularity f_0 is called *non-degenerate (in the Kouchnirenko sense) on a segment* $\gamma \in \Gamma(f_0)$ iff the system

$$\frac{\partial (f_0)_\gamma}{\partial x}(x,y) = 0 = \frac{\partial (f_0)_\gamma}{\partial y}(x,y)$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. f_0 is called *non-degenerate* iff it is non-degenerate on every segment $\gamma \in \Gamma(f_0)$.

For the sake of simplicity, we state the Kouchnirenko and Płoski Theorems only in the case of *convenient* singularities f_0, i.e. we demand $\Gamma_+(f_0)$ to intersect both coordinate axes Ox, Oy of $\mathbb{R}^2$. For such singularities we denote by $\mathcal{A}$ the area of the domain bounded by the coordinate axes and the Newton Polygon $\Gamma(f_0)$, while $\boldsymbol{a}$, (resp. $\mathbf{b}$) are: the distance of the point $(0,0)$ to the intersection of $\Gamma_+(f_0)$ with the Ox (resp. Oy) axis. The number

$$\nu(f_0) := 2\mathcal{A} - \boldsymbol{a} - \boldsymbol{b} + 1$$

is called the *Newton Number of the singularity* f_0. The following famous fact holds.

Theorem 1 (Kouchnirenko, [Kou76]) *For a convenient singularity* f_0 *it is:*

1. $\mu(f_0) \geqslant \nu(f_0)$,
2. *if* f_0 *is non-degenerate then* $\mu(f_0) = \nu(f_0)$.

Theorem 1 can be strengthen in the following way.

Theorem 2 (Płoski, [Pło90, Pło99]) *If for a convenient singularity* f_0 *it is* $\nu(f_0) = \mu(f_0)$ *then* f_0 *is non-degenerate.*

Remark. Under a suitable definition of the number $\nu(f_0)$, theorem 1 is also valid in the n–dimensional case. However, the theorem of Płoski is a purely 2–dimensional phenomenon; a suitable 3–dimensional example of a degenerate singularity f_0 for which $\nu(f_0) = \mu(f_0)$ was given in [Kou76, Remarque 1.21].

For a singularity f_0 we can consider *non-degenerate deformations of* f_0, that is such deformations (f_s) of f_0, that for small $|s| \neq 0$ the singularity f_s is non-degenerate. Then the smallest nonzero value among all the jumps of non-degenerate deformations of the singularity f_0 (cf. Section 1) will be called *the non-degenerate jump of the singularity* f_0 and denoted by $\lambda^{\mathrm{nd}}(f_0)$. In another words,

$$\lambda^{\mathrm{nd}}(f_0) := \min\left(\{\lambda((f_s)) : (f_s) \text{ – a non-degenerate deformation of } f_0\} \setminus \{0\}\right).$$

It turns out that this restricted jump of a singularity is possible to be determined in some important general cases using only elementary geometric-combinatorial methods. Namely, A. Bodin in [Bod07] (see also [Wal08], [Wal09], [Wal10], [Wal12] for a more complete exposition and some generalizations) managed to compute $\lambda^{\mathrm{nd}}(f_0)$ in the case of convenient singularities f_0 whose Newton Polygon is built of only one segment. Let, more precisely, $\Gamma(f_0) = \left\{\overline{(\boldsymbol{a},0)\,(0,\boldsymbol{b})}\right\}$ and let us put $d := \gcd(\boldsymbol{a},\boldsymbol{b})$. Then:

Theorem 3 (Bodin, [Bod07]) *Under the above assumptions and notations,*

a) *if* $d < \min(\boldsymbol{a},\boldsymbol{b})$ *then* $\lambda^{\mathrm{nd}}(f_0) = d$

b) *if* $d = \min(\boldsymbol{a},\boldsymbol{b})$ *then* $\lambda^{\mathrm{nd}}(f_0) = d - 1$.

The rest of the section is devoted mainly to the concept of a versal unfolding. It is based on the book by Ebeling [Ebe07]. Since we are not interested in the "semi-local" case, we adopt the definitions and the main result on versal unfoldings ([Ebe07, Prop. 3.17]) to the *local* situation.

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. An *unfolding of* f_0 is a holomorphic germ $F : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to (\mathbb{C}, 0)$ such that $F(z, 0) = f_0(z)$ and $F(0, u) = 0$.

Two unfoldings $F : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to (\mathbb{C}, 0)$ and $G : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to (\mathbb{C}, 0)$ of f_0 are said to be *equivalent*, if there exists a holomorphic map-germ

$$\psi : \left(\mathbb{C}^n \times \mathbb{C}^k, 0\right) \to (\mathbb{C}^n, 0), \quad \psi(z, 0) = z, \quad \psi(0, u) = 0$$

such that

$$G(z,u) = F(\psi(z,u), u).$$

It is easy to see that this notion of equivalence is in fact an equivalence relation in the set of unfoldings of f_0.

Let $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ be an unfolding of f_0 and $\varphi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ – a holomorphic map-germ. The *unfolding of f_0 induced from F by φ* is defined by the formula

$$G(z,u) = F(z, \varphi(u)).$$

An unfolding $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ of f_0 is called *versal* if any unfolding of f_0 is equivalent to one induced from F.

The following proposition will be useful.

Proposition 1 ([Mar82, Ch. 4, Prop. 2.4]) *If $f \in \mathcal{O}^n$ is an isolated singularity, $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}^n$, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}^n}{\mathfrak{m}(\nabla f)\mathcal{O}^n} = \dim_{\mathbb{C}} \frac{\mathcal{O}^n}{(\nabla f)\mathcal{O}^n} + n.$$

The main result concerning versal unfoldings is the following.

Theorem 4 *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a singularity and put $\mu = \mu(f_0)$. Let $g_1, \ldots, g_{\mu+n-1} \in \mathcal{O}^n$ be any representatives of a basis of the $\mathbb{C}$–vector space $\frac{\mathfrak{m}}{\mathfrak{m}(\nabla f_0)}$, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}^n$. Then the holomorphic germ*

$$F : (\mathbb{C}^n \times \mathbb{C}^{\mu+n-1}, 0) \to (\mathbb{C}, 0)$$

defined as

$$F(z,u) := u_1 g_1(z) + \ldots + u_{\mu+n-1} g_{\mu+n-1}(z) + f_0(z)$$

is a versal unfolding of f_0.

Remark. The proof of the above theorem runs in a very similar way to that given by Ebeling ([Ebe07, Prop. 3.17]; see also [Wal81, Thm. 3.4] for a more general, but less explicit, approach to the concept of a versal unfolding and a proof of Theorem 4).

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions. We say that *f is stably equivalent to g* (see [AGV85]) iff there exists $p \in \mathbb{N}, p \geqslant \max(m,n)$ such that

$$f(x_1, \ldots, x_n) + x_{n+1}^2 + \ldots + x_p^2 \underset{\substack{\textbf{bih.}\\ \textbf{equiv.}}}{\sim} g(y_1, \ldots, y_m) + y_{m+1}^2 + \ldots + y_p^2.$$

We note the following.

Proposition 2 *The jump of a singularity is an invariant of the stable equivalence.*

Proof. It is known, that the Milnor number is an invariant of stable equivalence. In particular, it easily follows that λ is a biholomorphic invariant. Thus, it suffices to prove that for a singularity $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and any $\boxed{p \geqslant n+1}$ it is

$$\lambda\left(f_0\left(x_1, \ldots, x_n\right)\right) = \lambda\left(f_0\left(x_1, \ldots, x_n\right) + x_{n+1}^2 + \ldots + x_p^2\right).$$

First note, that if (f_s) is a deformation of f_0 then the family

$$\left(f_s\left(x_1, \ldots, x_n\right) + x_{n+1}^2 + \ldots + x_p^2\right)$$

is a deformation of $f_0\left(x_1, \ldots, x_n\right) + x_{n+1}^2 + \ldots + x_p^2$ and by the above property of the Milnor number it is

$$\lambda\left(\left(f_s\left(x_1, \ldots, x_n\right)\right)\right) = \lambda\left(\left(f_s\left(x_1, \ldots, x_n\right) + x_{n+1}^2 + \ldots + x_p^2\right)\right).$$

It follows that $\lambda\left(f_0\left(x_1, \ldots, x_n\right)\right) \geqslant \lambda\left(f_0\left(x_1, \ldots, x_n\right) + x_{n+1}^2 + \ldots + x_p^2\right)$. We will prove that the opposite inequality also holds.

Let $x := (x_1, \ldots, x_p)$ and $x' := (x_1, \ldots, x_n)$. Put

$$\boxed{g_0\left(x\right) := f_0\left(x'\right) + x_{n+1}^2 + \ldots + x_p^2}.$$

Take any deformation (g_s) of the singularity g_0. One can assume that $\mu\left(g_s\right) < \mu\left(g_0\right)$ and $\mu\left(g_s\right) \neq 0$, i.e. the germs g_s are not smooth, for small $|s| \neq 0$. By Theorem 4, as a versal deformation of f_0 one can take

$$F\left(x', u\right) := u_1 h_1\left(x'\right) + \ldots + u_{\mu+n-1} h_{\mu+n-1}\left(x'\right) + f_0\left(x'\right),$$

where $\mu := \mu\left(f_0\right)$ and $h_1, \ldots, h_{\mu+n-1} \in \mathcal{O}^n$ constitute a basis of $\frac{\mathfrak{m}_n}{\mathfrak{m}_n(\nabla f_0)\mathcal{O}^n}$, $\mathfrak{m}_n$ denoting the maximal ideal of $\mathcal{O}^n$. Let, similarly, $\mathfrak{m}_p$ denote the maximal ideal of $\mathcal{O}^p \supset \mathcal{O}^n$. It is easy to see that

$$\left(\mathfrak{m}_n + \left(x_{n+1}, \ldots, x_p\right)\mathbb{C}\right)\mathcal{O}^n + \mathfrak{m}_p \cdot \left(\nabla f_0, x_{n+1}, \ldots, x_p\right)\mathcal{O}^p = \mathfrak{m}_p.$$

It follows that (the classes of) the elements of the set

$$\mathcal{B} := \{h_1, \ldots, h_{\mu+n-1}, x_{n+1}, \ldots, x_p\}$$

span the $\mathbb{C}$-linear space $\frac{\mathfrak{m}_p}{\mathfrak{m}_p(\nabla f_0, x_{n+1}, \ldots, x_p)\mathcal{O}^p}$. But $\left(\nabla f_0, x_{n+1}, \ldots, x_p\right)_{\mathcal{O}^p} = \left(\nabla g_0\right)_{\mathcal{O}^p}$. By Proposition 1, the set $\mathcal{B}$ is a basis of $\frac{\mathfrak{m}_p}{\mathfrak{m}_p(\nabla g_0)\mathcal{O}^p}$ since $\operatorname{card}\mathcal{B} = \mu + p - 1$ and $\mu = \mu\left(f_0\right) = \mu\left(g_0\right) = \dim_{\mathbb{C}} \frac{\mathcal{O}^p}{(\nabla g_0)\mathcal{O}^p}$. Thus the germ $G : \left(\mathbb{C}^p \times \mathbb{C}^{\mu+p-1}, 0\right) \to (\mathbb{C}, 0)$ given by

$$G\left(x, v\right) := v_1 h_1\left(x'\right) + \ldots + v_{\mu+n-1} h_{\mu+n-1}\left(x'\right) + v_{\mu+n} x_{n+1} + \ldots + v_{\mu+p-1} x_p + g_0\left(x\right)$$

is a versal unfolding of g_0. It means that for the deformation (g_s) one can find a holomorphic map-germ $\varphi : (\mathbb{C}, 0) \to \left(\mathbb{C}^{\mu+p-1}, 0\right)$ such that

$$g_s\left(\cdot\right) \underset{\substack{\textbf{bih.}\\ \textbf{equiv.}}}{\sim} G\left(\cdot, \varphi\left(s\right)\right),$$

for every small enough $|s| \neq 0$. But then $\mu(g_s) = \mu(G(\cdot, \varphi(s)))$ and since $G_{\varphi(s)} := G(\cdot, \varphi(s))$ is a deformation of g_0, also $\boxed{\lambda((g_s)) = \lambda\left(\left(G_{\varphi(s)}\right)\right)}$. Now, we assumed that the g_s'es were not smooth, so it has to be $\varphi_{\mu+n} = \ldots = \varphi_{\mu+p-1} = 0 \in \mathcal{O}$ or in another words

$$G_{\varphi(s)}(x) = \varphi_1(s) h_1(x') + \ldots + \varphi_{\mu+n-1}(s) h_{\mu+n-1}(x') + g_0(x).$$

Putting

$$h_s(x) := \varphi_1(s) h_1(x') + \ldots + \varphi_{\mu+n-1}(s) h_{\mu+n-1}(x') + f_0(x')$$

we have $G_{\varphi(s)}(x) - h_s(x) = x_{n+1}^2 + \ldots + x_p^2$, and so $\mu\left(G_{\varphi(s)}\right) = \mu(h_s)$, for small $|s| \neq 0$. Since (h_s) is a deformation of f_0 and $\mu(g_0) = \mu(f_0)$, it is $\lambda\left(\left(G_{\varphi(s)}\right)\right) = \lambda((h_s))$. Thus $\lambda((g_s)) = \lambda((h_s))$ and $\lambda(g_0) \geqslant \lambda(f_0)$. The proof is finished. □

3 Main Results

Since showing that $\lambda\left(x^4 + y^6 + bx^2y^4\right) = 1$ is much easier than proving that $\lambda\left(x^4 + y^4 + ax^2y^2\right) = 2$, we first address the first problem.

Theorem 5 *For the singularities $f_0(x,y) = x^4 + y^6 + bx^2y^4$, where $b \in \mathbb{C}$, it is*

$$\lambda(f_0) = 1.$$

In particular, $\lambda\left(x^4 + y^6\right) = 1$.

Proof. Fix any $b \in \mathbb{C}$. Since f_0 is Kouchnirenko non-degenerate, it follows that $\mu(f_0) = 15$. Consider the following deformation of f_0:

$$\boxed{f_s(x,y) := x^4 + \left(y^2 + sx\right)^3 + bx^2y^4.}$$

The deformation consists of degenerate singularities (for $s \neq 0$). Apply the following change of coordinates: $x \mapsto x - sy^2$,$y \mapsto sy$. In this coordinates the f_s'es take the form

$$\bar{f}_s(x,y) = s^3x^3 + (s^4 + bs^6)y^8 + \left[x^4 - 4sx^3y^2 + (6s^2 + bs^4)x^2y^4 - (4s^3 + 2bs^5)xy^6\right].$$

It is immediately seen that for $s \neq 0$ the singularities $\bar{f}_s$ are non-degenerate and so

$$\mu\left(\bar{f}_s\right) = 14.$$

Since the Milnor number is an invariant of a singularity, it is also

$$\mu(f_s) = 14.$$

It means that for this particular deformation (f_s) it is $\lambda((f_s)) = 1$. Therefore also $\lambda(f_0) = 1$, by the definition of the jump of a singularity. □

Remark. Theorem 3 (see also [Wal10, Corollary 2]) implies that for the above singularities f_0 their non-degenerate jumps are equal to 2.

We now present the proof of the main result of this work, namely that $\lambda\left(x^4+y^4+ax^2y^2\right)=2$. The proof, in part, was strongly supported by symbolic calculations (in the computer algebra system MAPLE).

Theorem 6 *For the singularities*

$$f_0(x,y)=x^4+y^4+ax^2y^2, \tag{1}$$

where $a\in\mathbb{C}, a^2\neq 4$, *it is*

$$\lambda(f_0)=2.$$

Thus for every singularity of type X_9 *its jump is equal to* 2.

First we state and prove two lemmas.

Lemma 1 *As a basis of the* $\mathbb{C}$*-vector space* $\mathfrak{m}/\,\mathfrak{m}(\nabla f_0)$, *where* $\mathfrak{m}$ *is the maximal ideal in* $\mathcal{O}^2$, *one can take the (classes of the) monomials* x^iy^j *with* $0<i+j\leqslant 3$ *and the monomial* x^2y^2.

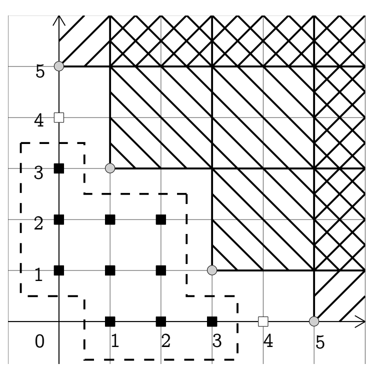
Proof of Lemma 1. Let us note that $\nabla f_0(x,y)=\left(4x^3+2axy^2, 4y^3+2ax^2y\right)$ and $x^5, x^3y\in\mathfrak{m}(\nabla f_0)$ in $\mathcal{O}^2$. Indeed, it is easy to check that

$$x^5=\left(\tfrac{x^2}{4}+\tfrac{2ay^2}{4(a^2-4)}\right)\frac{\partial f_0}{\partial x}+\left(\tfrac{-a^2xy}{4(a^2-4)}\right)\frac{\partial f_0}{\partial y}$$

and

$$x^3y=\left(\tfrac{-y}{(a^2-4)}\right)\frac{\partial f_0}{\partial x}+\left(\tfrac{ax}{2(a^2-4)}\right)\frac{\partial f_0}{\partial y}.$$

Since f_0 is symmetric with respect to x and y, also $y^5, xy^3\in\mathfrak{m}(\nabla f_0)$. Thus it is possible to depict the monomials that are potentially nonzero in $\mathfrak{m}/\,\mathfrak{m}(\nabla f_0)$ as follows:



We claim that the set $\mathcal{B}$ of the classes of the black points constitutes a basis of the $\mathbb{C}$-linear space $\mathfrak{m}/\ \mathfrak{m}\left(\nabla f_0\right)$. To see this, it is enough to note that $y^4 \equiv -\frac{a}{2}x^2y^2 \,(\mathrm{mod}\ \mathfrak{m}\nabla f_0)$, which means that $\overline{y^4} \in \mathrm{lin}_{\mathbb{C}}\,\mathcal{B}$ and by symmetry – also $\overline{x^4} \in \mathrm{lin}_{\mathbb{C}}\,\mathcal{B}$. Thus $\mathrm{lin}_{\mathbb{C}}\,\mathcal{B} = \mathfrak{m}/\ \mathfrak{m}\left(\nabla f_0\right)$. Since $\mu\left(f_0\right) = \dim_{\mathbb{C}} \mathcal{O}^2/\left(\nabla f_0\right) = 9$ and by Proposition 1 it is $\dim_{\mathbb{C}} \mathfrak{m}/\ \mathfrak{m}\left(\nabla f_0\right) = 10 = \mathrm{card}\,\mathcal{B}$, the set $\mathcal{B}$ is also linearly independent. □

Lemma 2 *For any complex numbers* $\mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ *with* $\mathfrak{h} \neq 0$ *the isolated singularity* $\mathfrak{F}$ *of the form*

$$\mathfrak{F}\left(x,y\right) = \left(x+y^2\right)^2 + \mathfrak{c}x^3 + \mathfrak{d}x^2y + \mathfrak{e}x^3y + \mathfrak{f}x^2y^2 + \mathfrak{g}xy^3 + \mathfrak{h}x^4 \tag{2}$$

has its Milnor number less than 8.

Proof of Lemma 2. Suppose that there exist complex numbers $\mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ such that $\mathfrak{h} \neq 0$ and the isolated singularity $\mathfrak{F}$ of the form (2) fulfills $\mu\left(\mathfrak{F}\right) \geqslant 8$. We compute the derivatives:

$$\frac{\partial \mathfrak{F}}{\partial x}\left(x,y\right) = 2\left(x+y^2\right) + 3\mathfrak{c}x^2 + 2\mathfrak{d}xy + 3\mathfrak{e}x^2y + 2\mathfrak{f}xy^2 + \mathfrak{g}y^3 + 4\mathfrak{h}x^3 \tag{3}$$

$$\frac{\partial \mathfrak{F}}{\partial y}\left(x,y\right) = 4y\left(x+y^2\right) + \mathfrak{d}x^2 + \mathfrak{e}x^3 + 2\mathfrak{f}x^2y + 3\mathfrak{g}xy^2. \tag{4}$$

Since $\mathrm{ord}_x \frac{\partial \mathfrak{F}}{\partial x} = 1$, it is possible to express the solution to the equation $\frac{\partial \mathfrak{F}}{\partial x}\left(\cdot, y\right) = 0$ as a function of y, namely $\frac{\partial \mathfrak{F}}{\partial x}\left(\varphi\left(y\right), y\right) = 0$ for the uniquely determined germ φ. Moreover, by the parametric definition of intersection multiplicity we have

$$8 \leqslant \mu\left(\mathfrak{F}\right) = i_0\left(\frac{\partial \mathfrak{F}}{\partial x}, \frac{\partial \mathfrak{F}}{\partial y}\right) = \mathrm{ord}_y \frac{\partial \mathfrak{F}}{\partial y}\left(\varphi\left(y\right), y\right). \tag{5}$$

Using (3) it is not hard to check that φ is of the following form

$$\varphi\left(y\right) = -y^2 - \frac{1}{2}\left(\mathfrak{g} - 2\mathfrak{d}\right)y^3 + \frac{1}{2}\left(\mathfrak{d}\mathfrak{g} - 2\mathfrak{d}^2 + 2\mathfrak{f} - 3\mathfrak{c}\right)y^4 + \ldots\ . \tag{6}$$

Taking into account (4) we conclude that the chunk of φ computed above allows us to correctly determine the terms of $\frac{\partial \mathfrak{F}}{\partial y}\left(\varphi\left(y\right), y\right)$ up to order 5 and (5) implies that these terms have to be equal to zero. Thus, substituting (6) into (4) and expanding, we arrive at

$$\mathrm{O}\left(y^8\right) = \frac{\partial \mathfrak{F}}{\partial y}\left(\varphi\left(y\right), y\right) = -5\left(\mathfrak{g} - \mathfrak{d}\right)y^4 + \frac{3}{2}\left(-\mathfrak{g}^2 + 4\mathfrak{d}\mathfrak{g} - 4\mathfrak{d}^2 + 4\mathfrak{f} - 4\mathfrak{c}\right)y^5 + \mathrm{O}\left(y^6\right).$$

The corresponding system of equations easily leads to the following unique set of relations:

$$\boxed{\mathfrak{d} = \mathfrak{g}, \mathfrak{c} = \tfrac{1}{4}(4\mathfrak{f} - \mathfrak{g}^2).} \tag{7}$$

Now we substitute (7) into (3) and (4):

$$\frac{\partial\mathfrak{F}}{\partial x}(x,y) = 2\left(x+y^2\right)+3(\mathfrak{f}-\frac{1}{4}\mathfrak{g}^2)x^2+2\mathfrak{g}xy+3\mathfrak{e}x^2y+2\mathfrak{f}xy^2+ +\mathfrak{g}y^3+4\mathfrak{h}x^3 \tag{8}$$

$$\frac{\partial\mathfrak{F}}{\partial y}(x,y) = 4y\left(x+y^2\right)+\mathfrak{g}x^2+\mathfrak{e}x^3+2\mathfrak{f}x^2y+3\mathfrak{g}xy^2 \tag{9}$$

and we compute the approximation of the expansion of φ a bit further:

$$\varphi(y) = -y^2+\frac{1}{2}\mathfrak{g}y^3-\frac{1}{8}(\mathfrak{g}^2+4\mathfrak{f})y^4-\frac{1}{4}\left(\mathfrak{g}^3-6\mathfrak{f}\mathfrak{g}+6\mathfrak{e}\right)y^5+\dots\,. \tag{10}$$

Substituting (10) into (9) we can find the expansion of $\frac{\partial\mathfrak{F}}{\partial y}(\varphi(y),y)$ up to order 6, namely

$$\mathrm{O}\left(y^8\right)=\frac{\partial\mathfrak{F}}{\partial y}(\varphi(y),y)=-\frac{7}{8}(\mathfrak{g}^3-4\mathfrak{f}\mathfrak{g}+8\mathfrak{e})y^6+\mathrm{O}\left(y^7\right).$$

The above equation leads to

$$\boxed{\mathfrak{e}=\tfrac{1}{8}\mathfrak{g}(4\mathfrak{f}-\mathfrak{g}^2).} \tag{11}$$

Using the relation (11) in (8) and (9) we get:

$$\frac{\partial\mathfrak{F}}{\partial x}(x,y) = 2\left(x+y^2\right)+\frac{3}{4}(4\mathfrak{f}-\mathfrak{g}^2)x^2+2\mathfrak{g}xy+\frac{3}{8}\mathfrak{g}(4\mathfrak{f}-\mathfrak{g}^2)x^2y+ +2\mathfrak{f}xy^2+\mathfrak{g}y^3+4\mathfrak{h}x^3$$

$$\frac{\partial\mathfrak{F}}{\partial y}(x,y) = 4y\left(x+y^2\right)+\mathfrak{g}x^2+\frac{1}{8}\mathfrak{g}(4\mathfrak{f}-\mathfrak{g}^2)x^3+2\mathfrak{f}x^2y+3\mathfrak{g}xy^2 \tag{12}$$

and then we compute the next term of φ, obtaining

$$\varphi(y) = -y^2+\frac{1}{2}\mathfrak{g}y^3-\frac{1}{8}(\mathfrak{g}^2+4\mathfrak{f})y^4-\frac{1}{16}\mathfrak{g}(\mathfrak{g}^2-12\mathfrak{f})y^5+ +\frac{1}{16}\left(\mathfrak{g}^4-4\mathfrak{f}\mathfrak{g}^2-16\mathfrak{f}^2+32\mathfrak{h}\right)y^6+\dots\,. \tag{13}$$

One last time we compute the approximation of $\frac{\partial\mathfrak{F}}{\partial y}(\varphi(y),y)$, this time using (13) in (12):

$$\mathrm{O}\left(y^8\right)=\frac{\partial\mathfrak{F}}{\partial y}(\varphi(y),y)=-\frac{1}{8}(\mathfrak{g}^4-8\mathfrak{f}\mathfrak{g}^2+16\mathfrak{f}^2-64\mathfrak{h})y^7+\mathrm{O}\left(y^8\right).$$

The above equation implies the following

$$\boxed{\mathfrak{h}=\tfrac{1}{64}\mathfrak{g}^4-\tfrac{1}{8}\mathfrak{f}\mathfrak{g}^2+\tfrac{1}{4}\mathfrak{f}^2=\tfrac{1}{64}(4\mathfrak{f}-\mathfrak{g}^2)^2.} \tag{14}$$

Putting $\boxed{\mathfrak{i}:=4\mathfrak{f}-\mathfrak{g}^2}$ we can sum up the relations (7), (11) and (14) as

$$\boxed{\mathfrak{d}=\mathfrak{g},\mathfrak{c}=\tfrac{1}{4}\mathfrak{i},\mathfrak{e}=\tfrac{1}{8}\mathfrak{g}\mathfrak{i},\mathfrak{h}=\tfrac{1}{64}\mathfrak{i}^2.} \tag{15}$$

Thus, written in terms of $\mathfrak{i}$ and $\mathfrak{g}$, $\mathfrak{F}$ takes the form

$$\begin{aligned}\mathfrak{F}(x,y) &= \left(x+y^2\right)^2+\frac{1}{4}\mathfrak{i}x^3+\mathfrak{g}x^2y+\frac{1}{8}\mathfrak{g}\mathfrak{i}x^3y+\frac{1}{4}(\mathfrak{i}+\mathfrak{g}^2)x^2y^2+\mathfrak{g}xy^3+\frac{1}{64}\mathfrak{i}^2x^4\\ &= \frac{1}{64}(8(x+y^2)+\mathfrak{i}x^2+4\mathfrak{g}xy)^2\end{aligned}$$

which is impossible, since $\mathfrak{F}$ is an isolated singularity. The lemma is proved. $\square$

Remark. By analyzing the proof of Lemma 2 and using Płoski Theorem, one can conclude that the singularities of the form (2) can have their Milnor numbers equal only to $4,5,6$ or 7.

Proof of Theorem 6. First note that it is enough to compute $\lambda(f_0)$ for f_0 of the form (1), or in another words for singularities being given in the normal form for the class X_9 (cf. [AGV85]), because — by Proposition 2 — the jump is an invariant of stable equivalence and each singularity of the family X_9 is stably equivalent to one of the form (1).

Let us fix $a \in \mathbb{C}, a^2 \neq 4$. We easily check that $\mu(f_0)=9$. Let us consider the deformation

$$\boxed{f_s(x,y) := x^4+(y^2+sx)^2+ax^2(y^2+sx).}$$

As was the case with Theorem 5, we apply now the change of coordinates: $x \mapsto x-sy^2$, $y \mapsto sy$, for $s \neq 0$. In this coordinates the f_s'es take the form

$$\bar{f}_s(x,y) = s^2x^2+as^3xy^4+s^4y^8+[asx^3+x^4-2as^2x^2y^2-4sx^3y^2+6s^2x^2y^4-4s^3xy^6].$$

It is easily seen that such $\bar{f}_s$'es are non-degenerate if $s\neq 0$ and $a \neq \pm 2$. Thus, by Kouchnirenko theorem, it is $\mu\left(\bar{f}_s\right)=\nu\left(\bar{f}_s\right)=7$ and so also

$$\mu(f_s) = 7 \text{ for } s \neq 0. \tag{16}$$

It means that $\lambda((f_s))=2$ and therefore $\lambda(f_0)\leqslant 2$. By the definition of the jump of a singularity, there are only two cases: $\lambda(f_0)=1$ or $\lambda(f_0)=2$. We will exclude the first possibility.

Suppose to the contrary, that there exists a deformation (f_s) of the singularity f_0 with the property that

$$\boxed{\mu(f_s) = 8 \text{ for } s \neq 0.} \tag{17}$$

By Theorem 4 it is possible to write the versal unfolding of f_0 as

$$\begin{aligned}f_{\mathfrak{S}}(x,y) &= s_{10}x+s_{01}y+s_{20}x^2+s_{11}xy+s_{02}y^2+s_{30}x^3+s_{21}x^2y+s_{12}xy^2+\\ &\quad +s_{03}y^3+s_{22}x^2y^2+f_0(x,y)\end{aligned}$$

and there exists a holomorphic mapping $\mathfrak{S}=(s_{10},\ldots,s_{22}):(\mathbb{C},0)\to\left(\mathbb{C}^{10},0\right)$ such that for every small enough $|s|\neq 0$ it is

$$f_s \underset{\substack{\textbf{bih.}\\ \textbf{equiv.}}}{\sim} f_{\mathfrak{S}(s)}.$$

It implies that $\mu(f_s) = \mu\left(f_{\mathfrak{S}(s)}\right)$ and so in the following we may assume that $f_s = f_{\mathfrak{S}(s)}$. Since $\mu(f_s) = 8 \neq 0$ for $s \neq 0$ then the germs f_s are not smooth. It follows that $\operatorname{ord} f_s \geqslant 2$ and that gives $s_{10}x + s_{01}y = 0$ or $s_{10} = s_{01} = 0$. Thus we have

$$(18)\; f_s(x,y) = s_{20}x^2 + s_{11}xy + s_{02}y^2 + s_{30}x^3 + s_{21}x^2y + s_{12}xy^2 + s_{03}y^3 + s_{22}x^2y^2 + f_0(x,y),$$

where $s_{ij}(0) = 0$.

From Theorem 3 it follows that the f_s'es have to be degenerate for small $|s| \neq 0$, so we can assume that this is the case for all f_s, $s \neq 0$. However, the singularity f_0 is non-degenerate and so we conclude by Płoski theorem 2 that it has to be

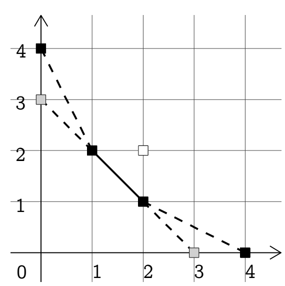
$$(19) \qquad \operatorname{ord} f_s < 4.$$

Thus we will distinguish two cases: $\operatorname{ord} f_s = 3$ and $\operatorname{ord} f_s = 2$. What is more, in the rest of the reasoning we choose and keep fixed *any* sufficiently small $s_0 \neq 0$.

I. $\operatorname{ord} f_{s_0} = 3$. That means we can write

$$f_{s_0}(x,y) = s_{30}x^3 + s_{21}x^2y + s_{12}xy^2 + s_{03}y^3 + (s_{22}+a)\,x^2y^2 + x^4 + y^4,$$

with $s_{ij} = s_{ij}(s_0) \in \mathbb{C}$. There are several options for the Newton diagram of f_{s_0}. However, f_{s_0} has to be degenerate, so the possibilities can be reduced to the following (the white point is optional, at least one of the grey points has to appear as a vertex of the diagram, and the black points are obligatory):



We will treat the above possibilities simultaneously. Namely, one can write them down in the following way

$$f_{s_0}(x,y) = (\alpha x + \beta y)^2 (\gamma x + \delta y) + (s_{22}+a)\,x^2y^2 + x^4 + y^4,$$

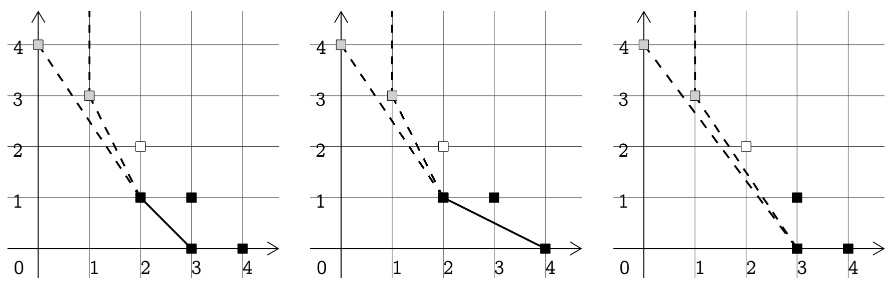
where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha\beta \neq 0$ and $(\gamma, \delta) \neq (0,0)$. Next we change the coordinates: $x \mapsto \frac{x}{\alpha}$, $y \mapsto \frac{y}{\beta}$ and after that f_{s_0} takes the form

$$\tilde{f}_{s_0}(x,y) = (x+y)^2 (\varepsilon x + \zeta y) + \rho x^2y^2 + \sigma x^4 + \tau y^4,$$

where $\sigma\tau \neq 0$ and $\varepsilon, \zeta \neq (0,0)$. We change the coordinates ones again: $x \mapsto x - y, y \mapsto y$ to obtain

$$\begin{aligned} \tilde{\tilde{f}}_{s_0}(x,y) &= \epsilon\, x^3 + (-\epsilon + \zeta)\, yx^2 + \sigma x^4 - 4\sigma\, yx^3 + (\rho + 6\,\sigma)\, y^2x^2 + \\ &\quad -2\,(\rho + 2\,\sigma)\, y^3x + (\sigma + \rho + \tau)\, y^4. \end{aligned}$$

Since $\operatorname{ord} \tilde{\tilde{f}}_{s_0} = 3$, the Newton diagram of $\tilde{\tilde{f}}_{s_0}$ is of one of the following forms (in each image the white point is optional, exactly one of the grey points has to appear as a vertex of the diagram, and the black points are obligatory):



In each of the above situations however, $\tilde{\tilde{f}}_{s_0}$ is easily seen to be non-degenerate and $\mu(\tilde{\tilde{f}}_{s_0}) = \nu(\tilde{\tilde{f}}_{s_0}) \leqslant 7$. Thus $\mu(f_{s_0}) \leqslant 7$, contradictory to (17).

II. $\operatorname{ord} f_{s_0} = 2$. Consider subcases

1. f_{s_0} is a reducible germ, or in another words we can write

$$f_{s_0} = f' f'', \qquad \operatorname{ord} f' = \operatorname{ord} f'' = 1.$$

Using the classical formula for the Milnor number of the product of two singularities (see e.g. [Cas00, Prop. 6.4.4]) we compute

$$\begin{aligned} 8 &= \mu(f_{s_0}) = (f' f'') = \mu(f') + 2\mu(f', f'') + \mu(f'') - 1 = \\ &= 2\mu(f', f'') - 1, \end{aligned}$$

which is impossible, $\mu(f', f'')$ being an integer.

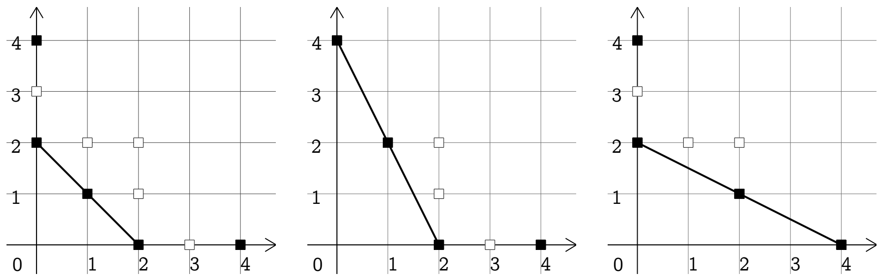
2. f_{s_0} is an irreducible germ. Since it is also a degenerate germ, it has to be of one of the following forms (cf. (18)):

   i. $f_{s_0}(x,y) = (\alpha x + \beta y)^2 +$ higher order terms, $\quad \alpha \neq 0, \beta \neq 0$,

   ii. $f_{s_0}(x,y) = \left(\alpha x + y^2\right)^2 +$ higher (weighted) order terms, $\quad \alpha \neq 0$,

   iii. $f_{s_0}(x,y) = \left(x^2 + \beta y\right)^2 +$ higher (weighted) order terms, $\quad \beta \neq 0$.

   More precisely, after taking (18) into account, one can sketch the Newton diagrams of f_{s_0} in each of the above cases, respectively as follows



Case (i) Case (ii) Case (iii)

Let us consider the case (i). Using (18) we can write

$$f_{s_0}(x,y) = (\alpha x+\beta y)^2 + s_{30}x^3 + s_{21}x^2y + s_{12}xy^2 + s_{03}y^3 + \\ + (s_{22}+a)\,x^2y^2 + x^4 + y^4,$$

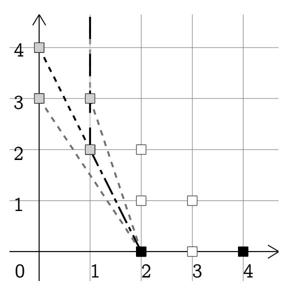
where $\alpha, \beta \in \mathbb{C}^*$ and $s_{ij} = s_{\mathrm{ij}}(s_0) \in \mathbb{C}$. If so, we have

$$f_{s_0}(x,y) = \alpha^2\left(x+\frac{\beta}{\alpha}y\right)^2 + s_{30}x^3 + s_{21}x^2y + s_{12}xy^2 + \\ + s_{03}y^3 + (s_{22}+a)\,x^2y^2 + x^4 + y^4, \tag{20}$$

and performing the change of coordinates $\mathcal{L} : x \mapsto x - \frac{\beta}{\alpha}y, y \mapsto y$ we are led to

$$\tilde{f}_{s_0}(x,y) := (f_{s_0}\circ\mathcal{L})(x,y) = \alpha^2x^2 + \text{middle terms} + x^4 + \\ + \left(1 + (s_{22}+a)\left(\frac{\beta}{\alpha}\right)^2 + \left(\frac{\beta}{\alpha}\right)^4\right)y^4. \tag{21}$$

The possible Newton diagrams of $\tilde{f}_{s_0}$ can be depicted as follows (the white points are optional, at least one of the grey points has to appear as a vertex of the diagram, and the black points are obligatory)



so, by Kouchnirenko theorem, $\tilde{f}_{s_0}$ has to be degenerate in order that $\mu\left(\tilde{f}_{s_0}\right) = 8$ (otherwise $\mu\left(\tilde{f}_{s_0}\right) = \nu\left(\tilde{f}_{s_0}\right) \leqslant 5 = \nu\left(x^2+xy^3\right)$ by the monotonicity of the Newton number with respect to Newton diagrams; cf. [Gwo08] or [Len08, Prop. 6.1]). But $\tilde{f}_{s_0}$ being degenerate implies that in fact there is only one possibility for the shape of $\tilde{f}_{s_0}$, namely (look at (21) and the figure above)

$$(22)\quad \tilde{f}_{s_0}(x,y) = \left(\alpha x + By^2\right)^2 + Cx^3 + Dx^2y + Ex^3y + Fx^2y^2 + Gxy^3 + x^4,$$

where $\alpha, \ldots, G \in \mathbb{C}$ and $\boxed{\alpha \neq 0 \neq B}$. We change the coordinates as follows: $x \mapsto \frac{x}{\alpha}$, $y \mapsto \frac{y}{\sqrt{B}}$, where $\sqrt{B} \in \mathbb{C}$ is a square root of $B \in \mathbb{C}$. In these new coordinates $\tilde{f}_{s_0}$ takes the form

$$\mathfrak{F}_{s_0}(x,y) = \left(x + y^2\right)^2 + \mathfrak{c}x^3 + \mathfrak{d}x^2y + \mathfrak{e}x^3y + \mathfrak{f}x^2y^2 + \mathfrak{g}xy^3 + \mathfrak{h}x^4,$$

where $\boxed{\mathfrak{h} \neq 0}$, and so Lemma 2 applies to $\mathfrak{F}_{s_0}$. As a consequence, $8 > \mu(\mathfrak{F}_{s_0}) = \mu(f_{s_0})$, which is contradictory to (17). This proves that the case (i) cannot happen.

Now we consider the second case. We see at once that if f_{s_0} is of the form (ii), it is in particular of the form (22) because $\alpha \neq 0$. It means that the reasoning carried on above for $\tilde{f}_{s_0}$ applies also to f_{s_0} of the form (ii) and so the case (ii) cannot happen.

The third case is immediately excluded by the symmetry of the indeterminates x and y in f_0.

Summing up, f_{s_0} cannot be an irreducible germ which means that (II) does not take place and thus $\operatorname{ord} f_{s_0} \neq 2$.

Since we have proved that f_{s_0} is neither of order 2 nor 3 and these are the only valid possibilities by (19), we arrive at a contradiction and thus we conclude that there is no deformation (f_s) of f_0 satisfying (17). On the other hand, we have indicated a deformation of f_0 with its jump equal to 2 (see (16)). By the definition of the jump of a singularity, the above means that $\lambda(f_0) = 2$. The proof is finished. □

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Skok liczb Milnora w klasie osobliwości X_9

Streszczenie. *Skok liczb Milnora osobliwości izolowanej* f_0 *to minimalna z niezerowych różnic pomiędzy liczbami Milnora osobliwości* f_0 *i jej deformacji* (f_s). *Dowodzimy, że dla osobliwości* $x^4+y^4+ax^2y^2$, *gdzie* $a \in \mathbb{C}, a^2 \neq 4$, *z klasy* X_9 *ich skok jest równy* 2.

Łódź, 9 – 13 stycznia 2012 r.