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ON EXTREME POINTS OF SOME SUBCLASSES OF CARATHÉODORY FUNCTIONS

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Let $\mathfrak{P}$ denote the class of functions of the form

(1) $$p(z) = 1 + p_1 z + \cdots p_n z^n + \cdots$$

holomorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$ ([1]).

Let us recall some properties of real parts of functions from $\mathfrak{P}$, which will be essential in the following.

(a) Every function $\operatorname{Re} p(z)$, $p \in \mathfrak{P}$, has Poisson representation by a unique positive measure μ (see [2], Chapter 1, p. 21–24)

(2) $$\operatorname{Re} p(z) = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

where $d\mu(t) \geq 0$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$.

(b) Let $d\mu(t) = f(t)\frac{dt}{2\pi} + d\sigma(t)$ be the Lebesgue decomposition of the representing measure μ with respect to the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $\langle -\pi, \pi \rangle$, i.e. $\int_{-\pi}^{\pi} f(t)\, dt < \infty$, $f \geq 0$, almost everywhere (a.e.) on $\langle -\pi, \pi \rangle$ with respect to $\frac{dt}{2\pi}$ and $d\sigma$ is singular. Then $\operatorname{Re} p(z)$ has nontangential limits $\operatorname{Re} p(e^{i\Theta})$ a.e. on $\langle -\pi, \pi \rangle$ and

(3) $$\operatorname{Re} p(e^{i\Theta}) = f(e^{i\Theta}) \qquad \text{a.e. on } \langle -\pi, \pi \rangle$$

(see [7], p. 29–30, Th. 5.3).

Now, we are in a position to give our main

Definition. Let $0 \le b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers and F_α be a given closed subset of the unit circle $T = \{z \in \mathbb{C}; |z| = 1\}$ of Lebesgue measure $2\pi\alpha$. Denote $\mathfrak{P}(B, b; F_\alpha)$ the class of functions $p \in \mathfrak{P}$ satisfying the following conditions:

(4) $$\operatorname{Re} p(e^{i\Theta}) \ge B \qquad \text{a.e. on } F_\alpha$$

and

(5) $$\operatorname{Re} p(e^{i\Theta}) \ge b \qquad \text{a.e. on } T \backslash F_\alpha.$$

The class $\mathfrak{P}(B, b; F_\alpha)$ has the following properties:

Theorem 1. *If $\mathfrak{P}(B, b; F_\alpha) \ne \emptyset$, then*

(6) $$1 \ge \alpha B + (1 - \alpha)\, b.$$

Proof. Let $p \in \mathfrak{P}(B, b; F_\alpha)$. Let $\omega(z; F_\alpha)$ be the harmonic measure of the set F_α with respect to $\mathbb{D}$ at the point z, i.e.

$$\omega(z; F_\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{F_\alpha} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt,$$

where χ_A is the characteristic function of the set A. Clearly, $0 < \omega(z; F_\alpha) < 1$ in $\mathbb{D}$ and by (3) $\omega(e^{it}; F_\alpha) = 1$ a.e. on F_α and $\omega(e^{it}; F_\alpha) = 0$ on $T \backslash F_\alpha$. Put $U_\alpha(z) = b + (B - b)\omega(z; F_\alpha)$. Then $U_\alpha(z) = B$ a.e. on F_α and $U_\alpha(z) = b$ on $T \backslash F_\alpha$. Since $\operatorname{Re} p(e^{i\Theta}) - U_\alpha(e^{i\Theta}) \ge 0$ a.e. on T, we have, for each $z \in \mathbb{D}$, by (2) and (3),

$$\begin{aligned}\operatorname{Re} p(z) &= \int_{-\pi}^{\pi} \operatorname{Re}(\frac{e^{it} + z}{e^{it} - z})\, d\mu(t) \ge \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} p(e^{it}) \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt \\ &\ge \frac{1}{2\pi} \int_{F_\alpha} B\, B \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt + \frac{1}{2\pi} \int_{T \backslash F_\alpha} b \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt = U_\alpha(z),\end{aligned}$$

hence

(7) $$\operatorname{Re} p(z) \ge b + (B - b)\omega(z; F_\alpha).$$

For $z = 0$ we obtain, with respect to (1) and $\omega(0; F_\alpha) = \alpha$, the inequality (6).

Theorem 2. *Let condition (2) hold. Then there exists a function $p_\alpha \in \mathfrak{P}(B, b; F_\alpha)$ so that $\operatorname{Re} p_\alpha(e^{i\Theta}) = B$ a.e.on F_α and $\operatorname{Re} p_\alpha(e^{i\Theta}) = b$ on $T \backslash F_\alpha$.*

Proof. Since $\mathbb{D}$ is simply connected, the function

(8) $$h(z; F_\alpha) = \omega(z; F_\alpha) + i\omega^*(z; F_\alpha) \qquad \text{where } \omega^*, \omega^*(0) = 0,$$

is the harmonic conjugate of ω, is by monodromy holomorphic in $\mathbb{D}$. Let there be equality in (6). Then $p_\alpha(z) = b + (B - b)h(z; F_\alpha)$, $z \in \mathbb{D}$, has the required property. If $B\alpha + b(1 - \alpha) < 1$, then the function $\tilde{p}_\alpha(z) = b + (B - b)h(z; F_\alpha)$ fulfils (4) and (5) but does not lie in $\mathfrak{P}(B, b; F_\alpha)$, since $\tilde{p}_\alpha(0) = B\alpha + b(1 - \alpha) < 1$. So it is natural to achieve the required normalization by adding a proper multiple of $\frac{e^{i\gamma} + z}{e^{i\gamma} - z}$, γ real. Clearly,

$$p_\alpha(z) = \tilde{p}_\alpha(z) + \{1 - [B\alpha + b(1 - \alpha)]\} \frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \quad z \in \mathbb{D}$$

is the required function.

Theorem 3. *The class $\mathfrak{P}(B,b;F_\alpha)$, where B, b, α satisfy condition (6), is compact in the topology given by the uniform convergence on compact subsets of $\mathbb{D}$.*

Denote by $\operatorname{Ext}(B,b;F_\alpha)$ the set of extreme points of $\mathfrak{P}(B,b;F_\alpha)$ and by $E(B,b;F_\alpha)$ the set of all $p \in \mathfrak{P}(B,b;F_\alpha)$ of the form

$$p(z) = b + (B-b)h(z;F_\alpha) + (1-\nu)\,\frac{e^{i\gamma}+z}{e^{i\gamma}-z}, \qquad z \in \mathbb{D} \tag{9}$$

where $h(z;F_\alpha)$ is function (8), γ is real and $\nu = B\alpha + b(1-\alpha)$. We have the following

Theorem 4. $\operatorname{Ext}(B,b;F_\alpha) = E(B,b;F_\alpha)$.

Let $I_\alpha \subset T$ be a given open arc of Lebesgue measure $2\pi\alpha$. In [3] the following subclass $\mathfrak{P}(B,b;I_\alpha)$ of $\mathfrak{P}$ was introduced: $p \in \mathfrak{P}(B,b;I_\alpha) \subset \mathfrak{P}$ if and only if

$$\liminf_{\substack{z\to z_0\\ z\in\mathbb{D}}} \operatorname{Re} p(z) \ge B \qquad \text{for each} \quad z_0 \in I_\alpha$$

and

$$\liminf_{\substack{z\to z_0\\ z\in\mathbb{D}}} \operatorname{Re} p(z) \ge b \qquad \text{for each} \quad z_0 \in T\backslash \bar{I}\alpha.$$

At the same time, we can consider the class $\mathfrak{P}(B,b;\bar{I}_\alpha)$ of functions from $\mathfrak{P}$ fulfilling (4) and (5). We have

Theorem 5. $\mathfrak{P}(B,b;I_\alpha) = \mathfrak{P}(B,b;\bar{I}_\alpha)$.

Corollary. *The set of extremal points of the class $\mathfrak{P}(B,b;I_\alpha)$ is the set $E(B,b;\bar{I}_\alpha)$ of functions of form (9).*

Theorem 6. *Let $p \in \mathfrak{P}(B,b;I_\alpha)$ has expansion (1) in $\mathbb{D}$, $\nu = \alpha B + (1-\alpha)\,b$. Then, for $n = 1,2,\dots$,*

$$|p_n| \le 2\Big[\frac{(B-b)|\sin n\alpha\pi|}{n\pi} + 1 - \nu\Big]. \tag{10}$$

Estimate (10) is sharp and is attained by function (9), where $e^{-in\gamma} = \operatorname{sgn}\sin n\alpha\pi$.

Proof. Clearly, we can confine ourselves to consider only the arc

$$I_\alpha = \{e^{i\varphi} \in T; -\alpha\pi < \varphi < \alpha\pi\}.$$

Since it is sufficient to verify the estimate (10) for extreme points of $\mathfrak{P}(B,b;I_\alpha)$ (see e.g. [8], p. 45, Th. 4.6), therefore, by Corollary, we have to write the Taylor expansion of the functions $\hat{p}$ of the form

$$\hat{p}(z) = b + (B-b)\int_{-\alpha\pi}^{\alpha\pi} \frac{e^{it}+z}{e^{it}-z}\,\frac{dt}{2\pi} + (1-\nu)\,\frac{e^{i\gamma}+z}{e^{i\gamma}-z}, \qquad z \in \mathbb{D}.$$

Since

$$\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-int}\, z^n, \qquad z \in \mathbb{D},$$

and, for $|z| < \rho < 1$, the series converges uniformly in $\langle -\pi, \pi \rangle$, we can integrate term by term. Because

$$\int_{-\alpha\pi}^{\alpha\pi} e^{-int}\, dt = \frac{2\sin n\alpha\pi}{n},$$

we obtain by elementary calculations

$$b + (B-b)\int_{-\alpha\pi}^{\alpha\pi} \frac{e^{it}+z}{e^{it}-z}\,\frac{dt}{2\pi} + (1-\nu)\,\frac{e^{i\gamma}+z}{e^{i\gamma}-Z} = 1 + 2\sum_{n=1}^{\infty}\Big[\frac{(B-b)\sin n\alpha\pi}{\pi n} + (1-\nu)\,e^{-in\gamma}\Big] z_n,$$

so

$$\hat{p}_n = 2\Big[\frac{(B-b)\sin n\alpha\pi}{\pi n} + (1-\nu)\,e^{-in\gamma}\Big].$$

By observing that the first term in the square bracket is real, we obtain (10).

The proofs of Theorems 3, 4 and 5 are contained in [4]. A more general case will be investigated in [5].

The idea of investigating the classes $\mathfrak{P}(B, b; I_\alpha)$ comes from paper [6] where similar subclasses of the known class $S(M)$ of bounded univalent functions were considered.

SPIS LITERATURY

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O punktach ekstremalnych pewnych podklas funkcji Carathéodory'ego

Streszczenie. Niech $\mathfrak{P}$ oznacza klasę funkcji $p(z) = 1 + p_1 z + \cdots p_n z^n + \cdots$ holomorficznych w kole jednostkowym $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ i spełniających

w tym kole warunek $\operatorname{Re} p(z) > 0$. Jak wiadomo, jeśli $p \in \mathfrak{P}$, to $\operatorname{Re} p(z)$ ma granicę kątową $\operatorname{Re} p(e^{i\Theta})$ p.w. na $\langle \pi, \pi \rangle$.

Niech $0 \leq b < 1$, $b < B$, $0 < \alpha < 1$ będą ustalonymi liczbami, zaś F_α danym domkniętym podzbiorem okręgu jednostkowego $T = \{z \in \mathbb{C} : |z| = 1\}$ o mierze Lebesgue'a $2\pi\alpha$. Oznaczmy przez $\mathfrak{P}(B, b; F_\alpha)$ klasę funkcji $p \in \mathfrak{P}$ spełniających warunki: $\operatorname{Re} p\, e^{i\Theta} \geq B$ prawie wszędzie na F_α oraz $\operatorname{Re} p\, e^{i\Theta} \geq b$ prawie wszędzie na $T \backslash F_\alpha$.

W pracy zbadano podstawowe własności klasy $\mathfrak{P}(B, b; F_\alpha)$. W szczególności wyznaczono zbiór punktów ekstremalnych tejże klasy. Otrzymane wyniki stanowią istotne rozszerzenie rezultatów zawartych w Materiałach XIII-ej Konferencji Szkoleniowej [3].

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