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ON THE CLOSURE OF THE CLASS $P(B, b, \alpha)$

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In the present article we continue paper [5] from the XV-th International Conference on Complex Analysis and Geometry. It belongs to the series of papers [1]–[5] where different classes of functions defined by conditions on the unit circle were studied.

For convenience of the reader, we repeat the definition of $P(B, b, \alpha)$. We shall denote by $\mathbb{C}$ the complex plane, by $\mathbb{D}$ and $\mathbb{T}$ the unit disc and the unit circle, repsectively. By P we denote the class of all functions p holomorphic and with positive real part in $\mathbb{D}$ whose Taylor expansion is

$$p(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots ,$$

and, finally, by $m(E)$ the Lebesgue measure (i.e. arc length measure) of any measurable subset of $\mathbb{T}$.

Definition 1. Let $0 \leq b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers. By $P(B, b, \alpha)$ we denote the class of functions $p \in P$ such that there exists a measurable set $F = F(p)$, $F \subset \mathbb{T}$, $m(F) = 2\pi\alpha$, such that

$$\text{Re}\, p(e^{i\Theta}) \geq B \quad \text{a. e. on } F \tag{1}$$

and

$$\text{(2)} \qquad \operatorname{Re} p(e^{i\Theta}) \geq b \quad \text{a. e. on } \mathbb{T}\backslash F.$$

Here $p(e^{i\Theta})$ are the nontangential limits of p which exist a. e. on $\mathbb{T}$.

In [5] (Th. 4) it was shown that: (a) $P(B, b, \alpha)$ is not convex, (b) $P(B, b, \alpha)$ is not compact, i.e. not closed, in the topology given by the uniform convergence on compact subsets of $\mathbb{D}$. The functions $p_n \in P(B, b, \alpha)$ which converge to a function $p_0 \in P$ not belonging to $P(B, b, \alpha)$ realize the maximum modulus of the n-th coefficient in the class $P(B, b, \alpha)$ ([3], Th. 8). So, the following three natural questions arise:

(a) Which are the compact subsets of $P(B, b, \alpha)$?
(b) What is the closure of $P(B, b, \alpha)$?
(c) What is the closed convex hull of $P(B, b, \alpha)$?

In this article we formulate the answers to these questions.

(a) Let $A \subset P(B, b, \alpha)$. Denote by A_0 the set of the characteristic functions of all sets $F \subset \mathbb{T}$, $m(F) = 2\pi\alpha$, on which (2) and (3) hold for some $p \in A$.

Theorem 1. *The subset $A \subset P(B, b, \alpha)$ is compact if and only if the set A_0 is closed in $L^1(\mathbb{T})$.*

This theorem was obtained by a thorough analysis of the proof of the compactness of the class $P(B, b, \alpha; F)$ (see Theorem 3 in [3]). Here $P(B, b, \alpha; F)$ is the class of functions $p \in P$ satisfying the following condition: there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that (1) and (2) are satisfied on $\mathbb{F}_\tau$ and $\mathbb{T}\backslash\mathbb{F}_\tau$, respectively ($F \subset \mathbb{T}$ is a measurable set of Lebesgue measure $m(F) = 2\pi\alpha$, $\mathbb{F}_\tau = \{\xi \in \mathbb{C};\ e^{-i\tau}\xi \in F\}$).

(b) **Theorem 2.** *The closure of $P(B, b, \alpha)$ is the set of all functions $p \in P$ which can be represented in the form*

$$p(z) = b + \frac{B-b}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{e^{it}+z}{e^{it}-z}dt + (1-\eta)g(z), \qquad z \in \mathbb{D}, \quad \eta = B\alpha + b(1-\alpha),$$

where f is an arbitrary measurable function on $\langle -\pi, \pi)$ with the following properties:

(i) $0 \leq f(t) \leq 1$ *a.e. on* $\langle -\pi, \pi)$,
(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \alpha$

and q is an arbitrary function in P.

Let us remark that in the definition of $P(B, b, \alpha)$ given in [5] it was required that the set F be, in addition, closed. For the validity of Theorems 1 and 2, the more general definition given in the present article is necessary.

(c) An immediate consequence of Theorem 2 is the following

Corollary. *The closed convex hull of $P(B, b, \alpha)$ is the same set as in Theorem 2.*

The proofs of Theorems 1 and 2 will be published elsewhere.

References

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O domknięciu klasy $P(B, b, \alpha)$

Streszczenie. Niech P oznacza znaną klasę funkcji $p(z) = 1 + q_1 z + \dots$ holomorficznych w kole jednostkowym $\mathbb{D}$ i takich, że $\operatorname{Re} p(z) > 0$ w $\mathbb{D}$. W artykule rozważane są zagadnienia (a), (b) i (c) dotyczące różnych własności podklasy $P(B, b, \alpha) \subset P$ określonej w Definicji 1. Otrzymane rezultaty stanowią rozszerzenie wyników z pracy [5].

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