# On the problem of stability of the multiplicity/degree of a polynomial along a family of basic semialgebraic sets

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# degree along a subset

**Question**

$X \subset \mathbb{R}^n$. *What is degree of* $f \in \mathbb{R}[X_1, \ldots, X_n]$ *along* $X$*?*

**Definition**

$$\deg^X f := \inf \left\{ \nu : \frac{f(x)}{|x|^\nu} \to 0 \textit{ as } X \ni x \to \infty \right\}.$$

**Consequence(s)**

$\deg^X f \leq \deg f$.

# local multiplicity along a subset

**Question**

*$X \subset \mathbb{R}^n$ and origin $\mathbf{0} \in \operatorname{clos}(X)$.*

*What is multiplicity of $f \in \mathbb{R}[X_1, \ldots, X_n]$ at $\mathbf{0}$ along $X$?*

**Definition**

$$\operatorname{mult}_{\mathbf{0}}^{X} f := \sup \left\{ \mu : \frac{f(x)}{|x|^{\mu}} \to 0 \text{ as } X \ni x \to \mathbf{0} \right\}.$$

**Consequence(s)**

$\operatorname{mult}_{\mathbf{0}}^{X} f \geq \deg \operatorname{in}(f)$.

# Degree = local multiplicity

Let $\iota : \mathbb{R}^n \setminus \mathbf{0} \to \mathbb{R}^n \setminus \mathbf{0}$ be $x \to \frac{x}{|x|^2}$.

Let $f \in \mathbb{R}[X_1, \ldots, X_n]$.

Thus $f = f_d + f_{d-1} + \ldots + f_0$ with $d = \deg f$.

Let $\tilde{f}(y) := |y|^{2d} f(\iota(y)) = f_d(y) + |y|^2 f_{d-1}(y) + \ldots + |y|^{2d} f_0$.

**Lemma**

*For $X \subset \mathbb{R}^n \setminus \mathbf{0}$ and $f \in \mathbb{R}[X_1, \ldots, X_n]$*

$$\deg^X f = 2d - \mathrm{mult}_{\mathbf{0}}^{\iota(X)} \tilde{f}.$$

# Semialgebraic setting

**Theorem (Folklore)**

*If $S \subset \mathbb{R}^n$ closed, semialgebraic and $\mathbf{0} \in S$, then*

$$\mathrm{mult}_{\mathbf{0}}^{S} f \in \mathbb{Q}_{\geq 0}$$

*for any $f \in \mathbb{R}[X_1, \ldots, X_n]$.*

# proof of Theorem "Folklore" I

Assume (for talk): $S =$ clos. of its inter.

Let $Z =$ Zariski clos. of $\partial S$.

Let $\mathbf{b}_0 : (M_0, E_0) \to (\mathbb{R}^n, \mathbf{0})$ blow.-up of $\mathbf{0}$.

Let $\pi := \sigma \circ \mathbf{b_0} : (M, E) \to (M_0, \sigma(E)) \to (\mathbb{R}^n, \pi(E))$ be an (admissible) emb. res. of sing. of $Z$, that is a *principalization* and *monomialization* of $I_Z$. Namely

# proof of Theorem "Folklore" II

- $E$ is a SNC divisor;
- $\pi^* I_Z$ is principal and monomial in $E$.

Let $\mathbf{m_0} \subset \mathbb{R}[X_1, \ldots, X_n]$ be max. ideal at $\mathbf{0}$.

Let $S^\pi := \mathrm{clos}(\pi^{-1}(S) \setminus E)$.

**Consequence(s)**

*1)* $E_\mathbf{0} := \pi^{-1}(\mathbf{0})$ *is SNC;*
*2)* $\pi^*(\mathbf{m_0})$ *is principal and monomial in* $E_\mathbf{0}$*;*
*3) For any comp.* $H$ *of* $E_\mathbf{0}$
*- either* $S^\pi \cap H$ *is Zariski dense in* $H$*,*
*- or is* $\emptyset$*.*

# proof of Theorem "Folklore" III

Let $E_{\mathbf{0}}^S :=$ union of comp. $H \subset E_{\mathbf{0}}$ s. t. $S^\pi \cap H$ is Zariski dense in $H$.

Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ and $H$ be a comp. of $E_{\mathbf{0}}^S$. Find

$$\pi^*((f)) = I_H^{\varphi_H} \cdot J$$

for $\varphi_H = \mathrm{mult}_H \pi^*(f) \in \mathbb{N}$, and $J$ ideal s. t. $Z(J) \cap H$ of codim $\geq 2$.

We get $\pi^*(\mathbf{m_0}) = I_H^{\alpha_H} \cdot M_H$ and deduce

$$\mathrm{mult}_{\mathbf{0}}^S f := \min_{H \in E_{\mathbf{0}}^S} \frac{\varphi_H}{\alpha_H}$$

# Motivations

Roughly speaking such investigations are related to (among other things):

- subalgebra of polynomials bounded on a given semialgebraic set (unbounded preferably);

- representation of positive polynomials;

- related problems to constrained optimization;

# what is it about

Let $f, g_1, \ldots, g_N \in \mathbb{R}[X_1, \ldots, X_n]$, pairwise $\mathbb{R}$-lin. indep. & both vanish. at $\mathbf{0}$.

Let $\underline{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N$.
Let $S_{\underline{t}} := \mathrm{clos}\{f_{\underline{t}} > 0\}$ for $\underline{t} \in \mathbb{R}^N$ with $f_{\underline{t}} = f + \sum_i t_i g_i$.

**Hypothesis:** $\{f_{\underline{t}} > 0\} \neq \emptyset$ *for all* $\underline{t}$.

**Problem (of Stability)**

*How many different functions* $\mathrm{mult}_{\mathbf{0}}^{S_{\underline{t}}}$ *are there ?*

# Statement of the result

**Theorem (Gr.& Michalska - 2016)**

*There exists $\Sigma \subset \mathbb{R}^N$ closed semialg. of codim $\geq 1$, s. t. for any con. comp. $\Lambda$ of $\mathbb{R}^N \setminus \Sigma$ and for any $\underline{t} \in \Lambda$ the fct. $\mathrm{mult}_{\mathbf{0}}^{S_{\underline{t}}}$ is indep. of $\underline{t}$.*

Known for $n = 2, N = 1$ from Kurdyka & Michalska & Spodzieja (2014).

# Ingredients of proof I

$N = 1$ and $g := g_1$.

Let $\pi : (M, E) \to (\mathbb{R}^n, \pi(E))$ be adm. resol. of sing. of both ideals $(f)$ and $(g)$ which we require to factor through the blowing-up of $\mathbf{0}$.

**Consequence(s)**

*$E_{\mathbf{0}} := \pi^{-1}(\mathbf{0})$ is SNC divisor.*

Theorem, for $N = 1$, reduces to show

**Lemma**

*There are at most fin. many $t$ for which exists a comp. $H_t$ of $E_{\mathbf{0}}$ s. t. $S_t^\pi \cap H_t$ is neither Zariski dense in $H_t$ nor empty, for $S_t^\pi := \mathrm{clos}(\pi^{-1}(S_t) \setminus E)$*

# ingredients of proof II

For $H$ a comp. of $E$ then

$$\pi^*((f)) = I_H^{\varphi_H} \cdot F_H,$$

and

$$\pi^*((g)) = I_H^{\gamma_H} \cdot G_H$$

and

$$\pi^*(\mathbf{m_0}) = I_H^{\alpha_H} \cdot M_H$$

$f(\mathbf{0}) = g(\mathbf{0}) = 0 \Longrightarrow \varphi_H, \gamma_H \geq \alpha_H \geq 1$ for each $H$ comp. of $E_{\mathbf{0}}$.

# ingredients of proof III - Illustration of typical argument

$H$ comp. of $E_{\mathbf{0}}$.

$\mathbf{p} \in H$ not a corner point of $E$. Nearby $\mathbf{p}$

$$\pi^* f = u^{\varphi_H} \cdot \psi_f, \quad \text{and} \quad \pi^* g = u^{\gamma_H} \cdot \psi_g$$

with $u$ loc. gen. of $I_H$ at $\mathbf{p}$, and $\psi_h$, for $h = f, g$, loc. unit at $\mathbf{p}$.

If $\frac{f}{g}|_H \neq const$, then $\pi^* f_t = u^{\min(\varphi_H, \gamma_H)} \cdot \psi_t$ near $\mathbf{p}$.

**Lemma**

$\min(\varphi_H, \gamma_H)$ *odd* $\Longrightarrow S_t^\pi \cap H = H$ *(Zariski dense in* $H$*).*