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ON A CLASS OF TYPICALLY-REAL FUNCTIONS IN THE HALF-PLANE

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As is known, in 1931 W. Rogosinski introduced the notion of typically-real functions. He also examined the basic properties of functions

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots , \qquad |z| < 1,$$

holomorphic and typically-real in the unit disc $|z| < 1$ ([7]). The class of such functions is most often denoted by T_R. Other properties of the class T_R were next the objects of interest of many mathematicians (e.g. [2], [6]).

Of late years, there have also been investigated various classes of holomorphic functions in the half-plane $\Pi^+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with the so-called hydrodynamic normalization (e.g. [1]). The aim of the present considerations is to study the basic properties of typically-real functions in the half-plane Π^+.

Let $H = H(\Pi^+)$ be the class of all functions f holomorphic in Π^+ such that

$$\lim_{\substack{z \to \infty \\ z \in \Pi^+}} (f(z) - z) = a \tag{1}$$

where a is some complex number.

Let $\mathcal{T}_R$ be a subclass of functions of H which take real values on the positive real half-axis only. It is the class of typically-real functions in Π^+ (cf. [7]).

Of course, if $f \in \mathcal{T}_R$, then we have $a = \bar{a}$ in normalization condition (1).

Analogously as in the class T_R one proves

Proposition 1. *If $f \in \mathcal{T}_R$, then, for $z \in \Pi^+$,*

$$\operatorname{Im} f(z) \begin{cases} > 0 & \textit{if} \quad \operatorname{Im} z > 0, \\ = 0 & \textit{if} \quad \operatorname{Im} z = 0, \\ < 0 & \textit{if} \quad \operatorname{Im} z < 0. \end{cases} \tag{2}$$

Moreover, each function $f \in H$ satisfying condition (2) is a function of the class $\mathcal{T}_R$.

We shall next prove

Theorem 1. *If f is a function of the class $\mathcal{T}_R$, then*

$$f^{(n)}(z) = \overline{f^{(n)}(z)} \qquad \textit{for} \quad z = \bar{z} \in \Pi^+, \quad n = 1, 2, \ldots . \tag{3}$$

Property (3) means that the function $f \in \mathcal{T}_R$ expands in some neighbourhood of each point $z = \bar{z} > 0$ in a Taylor series with centre at the point z and with real coefficients.

Proof. Take a function $f \in \mathcal{T}_R$. Let $s_* \in \Pi^+$ be any point such that $s_* = \bar{s}_* > 0$ and let

$$U_* = \{z \in \mathbb{C} : |z - s_*| < s_*\}. \tag{4}$$

In disc (4) we have

$$f(z) = f(s_*) + \sum_{n=1}^{\infty} \frac{f^{(n)}(s_*)}{n!}(z - s_*)^n, \qquad z \in U_*. \tag{5}$$

Let $z = \bar{z} \in U_*$. Then $f(z) = \overline{f(z)}$, and so, from (5) we get

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(s_*)}{n!}(z - s_*)^n = \sum_{n=1}^{\infty} \frac{\overline{f^{(n)}(s_*)}}{n!}(z - s_*)^n.$$

In view of the theorem on the uniqueness of an expansion of a function in a Taylor series,

$$f^{(n)}(s_*) = \overline{f^{(n)}(s_*)}, \qquad n = 1, 2, \ldots ,$$

which, in virtue of arbitrariness of the choice of $s_* = \bar{s}_* > 0$, proves the assertion of Theorem 1.

The proposition below is also true.

Proposition 2. *If $f \in \mathcal{T}_R$, then,*

$$\overline{f(z)} = f(\bar{z}), \qquad z \in \Pi^+. \tag{6}$$

Consequently, the image $f(\Pi^+)$ of the half-plane Π^+ is symmetric with respect to the real axis.

Proposition 2 is a simple consequence of the theorem on the uniqueness of an analytic extension and the Riemann-Schwarz symmetry principle ([5], p. 638, 673) for functions of the class $\mathcal{T}_R$.

Let us next notice that not all functions $f \in H$ satisfying (3) are of the class $\mathcal{T}_R$.

Example 1. Let

$$f_0(z) = z + a + z^{-2}, \qquad z \in \Pi^+, \quad a \in \mathbb{R}. \tag{7}$$

Of course, $f_0 \in H$ since f_0 is a holomorphic function in Π^+ and, by (7), we have (1). Moreover, for $z = \bar{z} > 0$, we have $f_0(z) = \overline{f_0(z)}$ and $f_0^{(n)}(z) = \overline{f_0^{(n)}(z)}$, $n = 1, 2, \ldots$. We can easily obtain that there exists $z_0 \in \Pi^+$, $z_0 \neq \bar{z}_0$, such that $f_0(z_0) = \overline{f_0(z_0)}$.

Summing up, we infer that $f_0 \notin \mathcal{T}_R$.

Example 2. A function of the form

$$f_1(z; a, b) = z + a + b/z, \qquad z \in \Pi^+, \quad a, b \in \mathbb{R}, \tag{8}$$

belongs to the class $\mathcal{T}_R$ if $b \leq 0$.

Similarly as in the case of the class T_R, the following theorem is true.

Theorem 2. *The class $\mathcal{T}_R$ is convex.*

Let us next denote by $\mathcal{S}_R$ the class of functions holomorphic and univalent in Π^+, satisfying the conditions

(i) $\lim_{\substack{z\to\infty \\ z\in\Pi^+}} (f(z) - z) = a$, $a \in \mathbb{R}$,

(ii) there exists a point $z_0 \in \Pi^+$, $z_0 = z_0(f)$, $z_0 = \bar{z}_0 > 0$, such that $f^{(n)}(z_0) = \overline{f^{(n)}(z_0)}$, $n = 0, 1, 2, \ldots$.

We can prove

Lemma 1. *If a function f is holomorphic in Π^+ and condition (ii) holds, then*

$$f^{(n)}(z) = \overline{f^{(n)}(z)} \qquad \textit{for} \quad z = \bar{z} > 0, \quad n = 0, 1, 2, \ldots. \tag{9}$$

From the definition of the class $\mathcal{S}_R$ and from Lemma 1 we obtain

Theorem 1′. *If $f \in \mathcal{S}_R$, then conditions (9) hold.*

The considerations carried out above also justify

Proposition 2′. *If $f \in \mathcal{S}_R$, then (6) hold, that is,*

$$\overline{f(z)} = f(\bar{z}), \qquad z \in \Pi^+.$$

Besides, Proposition 2′ implies

Theorem 3. *The inclusion $\mathcal{S}_R \subset \mathcal{T}_R$ takes place.*

Example 3. We can show that each function f_1 of form (8) for any $b \leq 0$ is a function of the class $\mathcal{S}_R$.

Example 4. A function of the form

$$f_2(z; a, b) = z + a + b/z^2, \qquad z \in \Pi^+, \quad a, b \in \mathbb{R},$$

belongs to the class $\mathcal{T}_R$ if $b \leq 0$, but do not belong to $\mathcal{S}_R$ when $b < 0$.

From the above examples we deduce, for instance, that

$$\mathcal{S}_R, \mathcal{T}_R \neq \emptyset, \qquad H \backslash \mathcal{T}_R \neq \emptyset, \qquad \mathcal{T}_R \backslash \mathcal{S}_R \neq \emptyset.$$

In turn, define the class $\overline{\mathcal{T}_R}$ of functions holomorphic in $\widetilde{\Pi} = \overline{\Pi^+} \backslash \{\infty\}$, satisfying condition (1) for $z \in \widetilde{\Pi}$ and taking real values on the non-negative real half-axis only, i.e. $f(z) = \overline{f(z)}$ if and only if $z = \bar{z}$, $z \in \widetilde{\Pi}$, and, moreover, such that $f(0) = 0$, $f'(0) > 0$.

The definitions of the classes $\mathcal{T}_R$ and $\overline{\mathcal{T}_R}$ imply the inclusion

$$\overline{\mathcal{T}_R} \subset \mathcal{T}_R,$$

and, what is more, the class $\overline{\mathcal{T}_R}$ is convex.

Functions of the class $\overline{\mathcal{T}_R}$ also satisfy the following property which is the equivalent of Proposition 1 for the class $\mathcal{T}_R$.

Proposition 3. *If $f \in \overline{\mathcal{T}_R}$, then, $z \in \widetilde{\Pi}$, we have*

$$\operatorname{Im} f(z) \begin{cases} > 0 & \text{when} \quad \operatorname{Im} z > 0, \\ = 0 & \text{when} \quad \operatorname{Im} z = 0, \\ < 0 & \text{when} \quad \operatorname{Im} z < 0. \end{cases}$$

Next, define the class $\overline{\mathcal{P}_R}$ of functions p holomorphic in $\widetilde{\Pi}$, satisfying the conditions

(10a) $$\lim_{\substack{z \to \infty \\ z \in \widetilde{\Pi}}} [z(p(z) - 1)] = a \in \mathbb{R},$$

(10b) $$\operatorname{Re} p(z) > 0, \qquad z \in \widetilde{\Pi},$$

(10c) $$p(0) \in \mathbb{R},$$

(10d) $$p^{(n)}(0) \in \mathbb{R}, \qquad n = 1, 2, \ldots .$$

Of course, this class is convex. Besides, from (10a) we get

$$\lim_{\substack{z \to \infty \\ z \in \widetilde{\Pi}}} p(z) = 1 =: p(\infty),$$

whereas from (10c), (10d) and Lemma 1

$$p^{(n)}(z) = \overline{p^{(n)}(z)}, \qquad z = \bar{z} \geq 0, \quad n = 0, 1, 2, \ldots .$$

What is more, we have the following

Theorem 4. *For any function $f \in \overline{\mathcal{T}_R}$, the function*

$$p(z) = \begin{cases} \frac{f(z)}{z} & \text{for} \quad z \in \overline{\Pi^+} \setminus \{0, \infty\}, \\ f'(0) & \text{for} \quad z = 0, \end{cases}$$

belongs to the class $\overline{\mathcal{P}_R}$.

Consider the converse problem. Let $p \in \overline{\mathcal{P}_R}$. Put

(11) $$f(z) = zp(z), \qquad z \in \widetilde{\Pi}.$$

Of course, f is holomorphic in $\widetilde{\Pi}$, $f(0) = 0$, $f'(0) = p(0) > 0$. From (10a) we obtain

$$\lim_{\substack{z \to \infty \\ z \in \widetilde{\Pi}}} (f(z) - z) = a \in \mathbb{R}.$$

Since

$$f(z) = z \left[p(0) + \sum_{n=1}^{\infty} \frac{p^{(n)}(0)}{n!} \, z^n \right]$$

is some disc $|z| < \delta$, therefore from (10c) and (10d) we get that $f(z) = \overline{f(z)}$ for $z = \bar{z}$, $|z| < \delta$. In this disc we have

$$f'(z) = p(0) + \sum_{n=1}^{\infty} \frac{p^{(n)}(0)}{n!} \, (n+1) z^n$$

and, generally,

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{p^{(n-1)}(0)}{(n-1)!} n(n-1)(n-2)\cdots(n-m+1)z^{n-m}, \qquad m = 1,2,\dots .$$

In view of (10c) and (10d), condition (ii) is satisfied at each point $z_0 = \bar{z}_0$ of the disc $|z| < \delta$. In virtue of Lemma 1, the function f satisfies conditions (9).

It remains to show that $f(z) = \overline{f(z)}$ only if $z = \bar{z} \in \widetilde{\Pi}$. From (11) and (10b) we infer that, for $z = iy$, $y \neq 0$,

$$\operatorname{Re} p(z) = \operatorname{Re} \frac{f(iy)}{iy} = \frac{\operatorname{Im} f(iy)}{y} > 0.$$

So, if $y > 0$, then $\operatorname{Im} f(iy) > 0$, whereas if $y < 0$, then $\operatorname{Im} f(iy) < 0$. On the real half-axis $z \geq 0$ we have $\operatorname{Im} f(z) = 0$. By condition (1), for z tending to infinity, $\operatorname{Im} f(z)$ has the same sign as $\operatorname{Im} z$. Hence, applying the minimum principle to the function $\operatorname{Im} f(z)$, $z \in \{z \in \mathbb{C} : \operatorname{Im} z \geq 0 \wedge \operatorname{Re} z \geq 0\}$, we deduce that $\operatorname{Im} f(z) > 0$ when $\operatorname{Im} z > 0$ and $z \in \Pi^+$. In an analogous way we infer that $\operatorname{Im} f(z) < 0$ when $z \in \Pi^+$, $\operatorname{Im} z < 0$.

We have thus proved

Theorem 4′. *For any function $p \in \overline{\mathcal{P}_R}$, function (11) belongs to the class $\overline{\mathcal{T}_R}$.*

Example 5. A function of the form

$$f_3(z; a, b, \delta) = z + a + b/(z+\delta), \quad z \in \widetilde{\Pi}, \quad a, b, \delta \in \mathbb{R}, \quad b \leq 0, \quad \delta > 0, \quad a\delta + b = 0,$$

belongs to the class $\overline{\mathcal{T}_R}$.

Example 6. In view of Example 5 and according to Theorem 4, functions of the form

$$p(z; a, b, \delta) = \begin{cases} 1 + \frac{a}{z} + \frac{b}{z(z+\delta)} & \text{for} \quad z \in \overline{\Pi^+} \setminus \{0, \infty\}, \\ 1 - \frac{b}{\delta^2} & \text{for} \quad z = 0, \end{cases}$$

where $a, b, \delta \in \mathbb{R}$, $b \leq 0$, $\delta > 0$, $a\delta + b = 0$ belong to the class $\overline{\mathcal{P}_R}$.

Let us come back to the classes $\mathcal{S}_R$, $\mathcal{T}_R$. We can prove

Theorem 5. *The families $\mathcal{S}_R$, $\mathcal{T}_R$ are not compact.*

There are also other properties of the class $\mathcal{T}_R$. In particular, we have

Theorem 6. *If a function of the class $\mathcal{T}_R$ satisfies the condition*

$$\lim_{\substack{z\to 0\\ z=\bar{z}>0}} f(z) = 0,$$

then

$$\mathrm{Re}\,\frac{f(z)}{z} > 0, \qquad z \in \Pi^+.$$

To finish with, let us observe that some of the properties of the class T_R is easily carried over to the class $\mathcal{T}_R$. Other ones get complicated distinctly. The main assertions of the paper appeared in [3]. The omitted proofs and other properties of the classes of functions being investigated are in press ([4]).

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O klasie funkcji typowo-rzeczywistych w półpłaszczyźnie $\mathrm{Re}\,z > 0$

Streszczenie. Jak wiadomo, W. Rogisinski w 1931 roku wprowadził pojęcie funkcji typowo-rzeczywistych. Zbadał on również podstawowe własności funkcji

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots\,, \qquad |z| < 1,$$

holomorficznych i typowo-rzeczywistych w kole jednostkowym $|z| < 1$ ([7]). Klasa takich funkcji najczęściej oznaczana jest przez T_R. Inne własności klasy T_R stanowiły następnie przedmiot zainteresowań wielu matematyków (np. [2], [6]).

W ostatnich latach badane są też różne klasy funkcji holomorficznych w półpłaszczyźnie $\mathrm{Re}\,z > 0$ z tzw. normalizacją hydrodynamiczną (np. [1]).

Celem niniejszych rozważań jest zbadanie podstawowych własności funkcji typowo-rzeczywistych w wyżej wymienionej półpłaszczyźnie.

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