MATERIAŁY XV KONFERENCJI SZKOLENIOWEJ
Z ANALIZY I GEOMETRII ZESPOLONEJ

1994 **Łódź** **str. 31**

ON CERTAIN CLASSES OF QUASI-TYPICALLY-REAL FUNCTIONS

Z.J. Jakubowski (Łódź), A. Lecko (Rzeszów)

1. Let $\mathbb{C}$ denote the complex plane, $U_r = \{z \in \mathbb{C};\ |z| < r\}$, $r > 0$, the disc of radius r, $U = U_1$ the unit disc.

Definition 1.1. Denote by $\tilde{H}_\alpha$, $\alpha \in \mathbb{C}$, $|\alpha| \le 1$, the class of functions of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \tag{1.1}$$

which are regular in U and satisfy the following condition:

$$\operatorname{Re}\left\{(1 - \alpha^2 z^2)\frac{f(z)}{z}\right\} > 0, \qquad z \in U. \tag{1.2}$$

By condition (1.2) we see that the condition $|\alpha| \le 1$ is natural. The assumption $|\alpha| > 1$ implies that $\tilde{H}_\alpha = \emptyset$.

Let now $\alpha = |\alpha| e^{i\Theta}$, $\Theta \in \mathbb{R}$, $|\alpha| \le 1$. If $f \in \tilde{H}_\alpha$, then the function $g(z) = e^{i\Theta} f(e^{-i\Theta} z)$, $z \in U$, belongs to the class $\tilde{H}_{|\alpha|}$ and conversely. Therefore we will assume that $\alpha \in \langle 0, 1\rangle$.

Definition 1.2. Denote by $H_\alpha \subset \tilde{H}_\alpha$ the class of regular functions of form (1.1) for which $a_n \in \mathbb{R}$, $n = 2, 3, \ldots$.

Let P denote the well-known class of regular functions p in U of the form $p(z) = 1 + p_1 z + \ldots + p_n z^n \ldots$, having positive real part in U and $P_R \subset P$ the class of functions for which p_n, $n \in \mathbb{N}$, are real.

Condition (1.2) is equivalent to the following condition: $f \in \tilde{H}_\alpha$ (or H_α) if and only if there exists the function $p \in P$ (or P_R) such that

$$f(z) = \frac{z}{1-\alpha^2 z^2} p(z), \qquad z \in U, \tag{1.3}$$

or for $\alpha > 0$

$$f(z) = \frac{h(z)}{1-\alpha^2 z^2}, \qquad z \in U, \quad h \in \tilde{H}_0 \ (H_0).$$

If we take $\alpha = 1$ in condition (1.2), then we obtain the class $\tilde{H}_1$ examined by W. Hengartner and G. Schober in [2]. On the other hand the class H_1 is identical with the known class T_R typically-real functions of form (1.1), ([6]). For $\alpha = 0$ we have the class $\tilde{H}_0$ that contains functions $zp(z)$, $z \in U$, where $p \in P$.

The following theorem says that the classes $\tilde{H}_\alpha$ are really different among them.

Theorem 1.1. *If* $\alpha \in \langle 0, 1\rangle$ *and* $\beta \in \langle 0, \alpha) \cup (\alpha, 1\rangle$, *then* $\tilde{H}_\alpha \not\subseteq \tilde{H}_\beta$ *and* $\tilde{H}_\beta \not\subseteq \tilde{H}_\alpha$.

Proof. The proof consists of two cases:

1) Let $0 < \alpha \leq 1$, $0 \leq \beta < \alpha$ and f be the following

$$f(z) = \frac{z}{1-\alpha^2 z^2} \frac{1+z}{1-z}, \qquad z \in U.$$

Then $f \in \tilde{H}_\alpha$ and moreover,

$$(1-\beta^2 z^2)\frac{f(z)}{z} = \frac{1-\beta^2 z^2}{1-\alpha^2 z^2} \frac{1+z}{1-z}.$$

Let $z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Therefore $\frac{1+z_0}{1-z_0} = (\sqrt{2}+1)i$ and

$$\frac{1-\beta^2 z_0^2}{1-\alpha^2 z_0^2} \frac{1+z_0}{1-z_0} = \left(\frac{1+\alpha^2\beta^2}{1+\alpha^4} i - \frac{\alpha^2-\beta^2}{1+\alpha^4}\right)(\sqrt{2}+1). \tag{1.4}$$

We choose the sequence $(z_n)_{n=1}^{\infty}$ such that $z_n \in U$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = z_0$. Then, from (1.4), holds

$$\lim_{n\to\infty} \operatorname{Re}\left(\frac{1-\beta^2 z_n^2}{1-\alpha^2 z_n^2} \frac{1+z_n}{1-z_n}\right) = \operatorname{Re}\left(\frac{1-\beta^2 z_0^2}{1-\alpha^2 z_0^2} \frac{1+z_0}{1-z_0}\right) < 0.$$

Thus for suficiently large n we finally obtain

$$\operatorname{Re}\left(\frac{1-\beta^2 z_n^2}{1-\alpha^2 z_n^2} \frac{1+z_n}{1-z_n}\right) < 0.$$

This means that $f \notin \tilde{H}_\beta$, so $\tilde{H}_\alpha \not\subseteq \tilde{H}_\beta$.

2) Let $0 \leq \alpha < 1$, $\alpha < \beta \leq 1$ and f be the following

$$f(z) = \frac{z}{1-\alpha^2 z^2} \frac{1-z}{1+z}, \qquad z \in U.$$

Thus $f \in \tilde{H}_\alpha$. Let again $z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Then

$$\frac{1-\beta^2 z_0^2}{1-\alpha^2 z_0^2}\frac{1+z_0}{1-z_0} = \left(\frac{\alpha^2-\beta^2}{1+\alpha^4} - i\frac{1+\alpha^2\beta^2}{1+\alpha^4}\right)(\sqrt{2}-1).$$

Arguing so as in part 1 we have that $f \notin \tilde{H}_\beta$, so $\tilde{H}_\alpha \not\subseteq \tilde{H}_\beta$.

The proof for the class H_α is similar.

2. The next theorems give the integral representation for the classes $\tilde{H}_\alpha$ and H_α for $\alpha \in \langle 0, 1\rangle$.

Let $M(0, 2\pi)$ denote the set of all functions $\mu : \langle 0, 2\pi\rangle \to \mathbb{R}$ that are nondecreasing in the interval $\langle 0, 2\pi\rangle$ and satisfy the condition $\int_0^{2\pi} d\mu(t) = 2\pi$.

Theorem 2.1. *A function f belongs to the class $\tilde{H}_\alpha$, $\alpha \in \langle 0, 1\rangle$, if and only if*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} K_1(z,t)d\mu(t), \qquad z \in U, \tag{2.1}$$

where $\mu \in M(0, 2\pi)$ and

$$K_1(z,t) = \frac{z}{1-\alpha^2 z^2}\frac{1+ze^{-it}}{1-ze^{-it}}, \qquad z \in U, \quad t \in \langle 0, 2\pi\rangle. \tag{2.2}$$

Theorem 2.2. *A function f belongs to the class H_α, $\alpha \in \langle 0, 1\rangle$, if and only if*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} K_2(z,t)d\mu(t), \qquad z \in U, \tag{2.3}$$

where $\mu \in M(0, 2\pi)$ and

$$K_2(z,t) = \frac{1-z^2}{1-\alpha^2 z^2}K(z,t) = \frac{1-z^2}{1-\alpha^2 z^2}\frac{z}{1-2z\cos t + z^2}, \tag{2.4}$$

$z \in U$, $t \in \langle 0, 2\pi\rangle$.

Of course, Theorems 2.1 and 2.2 follow immediately from the Herglotz representation formula for the classes P and P_R ([3]) and from (1.3).

Obviously, for $\alpha = 1$ we have from (2.4) the integral representation for the class T_R ([5]).

The kernels $K_1(z,t)$ and $K_2(z,t)$ defined in (2.2) and (2.4) are regular in U and continuous in $\langle 0, 2\pi\rangle$. Let now $z \in U$ be a fixed point. Therefore integrals (2.1) and (2.3) are a weighted averages of the points on the curves $\Gamma_1 : w = K_1(z,t)$ and $\Gamma_2 = K_2(z,t)$ where $t \in \langle 0, 2\pi\rangle$, respectively. Thus from Theorems 2.1 and 2.2 we obtain

Corollary 2.1. *Let $z \in U$ be a fixed point. Then the region $\tilde{\mathbb{D}}_\alpha$ of values of the functional $G(f) = f(z)$, $f \in \tilde{H}_\alpha$, $\alpha \in \langle 0, 1\rangle$, is the closed convex hull of the curve $\Gamma_1 : w = K_1(z,t)$, $t \in \langle 0, 2\pi\rangle$.*

Corollary 2.2. *Let $z \in U$, $\mathrm{Im}(z) \neq 0$, be a fixed point. Then the region $\mathbb{D}_\alpha$ of values of the functional $G(f) = f(z)$, $f \in \tilde{H}_\alpha$, $\alpha \in \langle 0, 1\rangle$, is the closed convex hull of the curve $\Gamma_2 : w = K_2(z, t)$, $t \in \langle 0, 2\pi\rangle$. If $\mathrm{Im}(z) = 0$, then $\mathbb{D}_\alpha$ is the closed segment of the real axis having end-points at $K_2(z, \pi)$ and $K_2(z, 0)$.*

3. Using (2.1) and (2.3) we can find the formulae for coefficients of functions of the classes $\tilde{H}_\alpha$ and H_α.

Theorem 3.1. *If $f \in \tilde{H}_\alpha$, $\alpha \in \langle 0, 1)$, and if f is of form (1.1), then*

$$a_{2k} = \frac{1}{\pi}\int_0^{2\pi} e^{-(2k-1)it}\frac{(\alpha e^{it})^{2k} - 1}{(\alpha e^{it})^2 - 1}d\mu(t),$$
$$a_{2k+1} = \frac{1}{\pi}\int_0^{2\pi}\left(e^{-2kit}\frac{(\alpha e^{it})^{2(k+1)} - 1}{(\alpha e^{it})^2 - 1} - \frac{1}{2}\alpha^2\right)d\mu(t)$$

for $k \in \mathbb{N}$, $\mu \in M(0, 2\pi)$.

Remark 3.1. If $f \in \tilde{H}_1$, and if f is of form (1.1), then

$$a_{2k} = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin kt}{\sin t}e^{-ikt}d\mu(t),$$
$$a_{2k+1} = -1 + \frac{1}{\pi}\int_0^{2\pi}\frac{\sin(k+1)t}{\sin t}e^{-ikt}d\mu(t),$$

$k \in \mathbb{N}$, $\mu \in M(0, 2\pi)$.

Theorem 3.2. *If $f \in H_\alpha$, $\alpha \in \langle 0, 1)$, and if f is of form (1.1), then*

$$a_{2k} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin 2kt}{\sin t}d\mu(t) + (\alpha^2 - 1)\frac{1}{2\pi}\int_0^{2\pi}\frac{\alpha^{2k}\sin 2t - \alpha^2\sin 2kt + \sin 2(k-1)t}{(\alpha^4 - 2\alpha^2\cos 2t + 1)\sin t}d\mu(t),$$

$$a_{2k+1} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(2k+1)t}{\sin t}d\mu(t) + (\alpha^2 - 1)\frac{1}{2\pi}\int_0^{2\pi}\frac{(\alpha^{2k} + \alpha^{2k+2})\sin t - \alpha^2\sin(2k+1)t + \sin(2k-1)t}{(\alpha^4 - 2\alpha^2\cos 2t + 1)\sin t}d\mu(t),$$

$k \in \mathbb{N}$, $\mu \in M(0, 2\pi)$.

Remark 3.2 ([5]). If $f \in T_R$, and if f is of form (1.1), then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin nt}{\sin t}d\mu(t), \qquad n \in \mathbb{N}, \quad \mu \in M(0, 2\pi).$$

Now, we get the theorem concerning the estimations of all coefficients of function (1.1) from the class $\tilde{H}_\alpha$ or H_α.

Theorem 3.3. *If $f \in H_\alpha$, $\alpha \in \langle 0,1 \rangle$, and if f is of form (1.1), then*

$$|a_{k+1} - \alpha^2 a_{k-1}| \le 2, \qquad (a_0 = 0, \quad a_1 = 1), \tag{3.1}$$

$$|a_{2k}| \le \begin{cases} 2\frac{1-\alpha^{2k}}{1-\alpha^2}, & \alpha \in \langle 0,1), \\ 2k & \alpha = 1, \end{cases} \tag{3.2}$$

$$|a_{2k+1}| \le \begin{cases} \frac{2-\alpha^{2k}-\alpha^{2(k+1)}}{1-\alpha^2}, & \alpha \in \langle 0,1), \\ 2k+1 & \alpha = 1, \end{cases} \tag{3.3}$$

where $k \in \mathbb{N}$. The equalities in (3.1), (3.2), (3.3) are realized by the function

$$f_\alpha(z) = \frac{z}{1-\alpha^2 z^2}\frac{1+\varepsilon z}{1-\varepsilon z}, \qquad z \in U, \quad \varepsilon \in \mathbb{C}, \quad |\varepsilon| = 1. \tag{3.4}$$

Remark 3.3. We can also obtain a theorem for the class H_α similar to Theorem 3.3

4. The distortion theorem for $\tilde{H}_\alpha$ is the following

Theorem 4.1. *If $f \in \tilde{H}_\alpha$, $\alpha \in \langle 0,1 \rangle$, then*

$$\frac{r(1-r)}{(1+r)(1+\alpha^2 r^2)} \le |f(z)| \le \frac{r(1+r)}{(1-r)(1-\alpha^2 r^2)}, \tag{4.1}$$

$$|f'(z)| \le \frac{1+2r-(1-\alpha^2)r^2-2\alpha^2 r^3-\alpha^2 r^4}{(1-\alpha^2 r^2)^2(1-r)^2} \tag{4.2}$$

for $|z| \le r < 1$.

Remark 4.1. Estimations (4.1) and (4.2) are sharp. The upper bounds are achieved by the function f_α, $\varepsilon = 1$, (defined in (3.4)) at the point $z = r$. The lower bound in (4.1) is realized by the function f_α, $\varepsilon = i$, at the point $z = ir$.

From Corollary 2.2 we can obtain the upper bound for $f(z)|$ for the class H_α.

Theorem 4.2. *If $f \in H_\alpha$, $\alpha \in \langle 0,1 \rangle$, then*

$$|f(z)| \le \begin{cases} \left|\frac{z(1+z)}{(1-z)(1-\alpha^2 z^2)}\right|, & \textit{if} \ \ \mathrm{Re}\left(z+\frac{1}{z}\right) \ge 2, \\ \left|\frac{1-z^2}{1-\alpha^2 z^2}\right| \left|\mathrm{Im}\left(z+\frac{1}{z}\right)\right|^{-1}, & \textit{if} \ \ \left|\mathrm{Re}\left(z+\frac{1}{z}\right)\right| \le 2, \\ \left|\frac{z(1-z)}{(1+z)(1-\alpha^2 z^2)}\right|, & \textit{if} \ \ \mathrm{Re}\left(z+\frac{1}{z}\right) \le -2, \end{cases} \tag{4.3}$$

where $z \in U$, $z \ne 0$.

Of course, from Theorem 4.2 we obtain immediately the known estimations $|f(z)|$ for the functions $f \in T_R$, ([1]).

Theorem 4.3. *If $f \in H_\alpha$, $\alpha \in \langle 0,1\rangle$, then*

$$\operatorname{Arg}\frac{1-z}{(1+z)(1-\alpha^2z^2)} \le \operatorname{Arg}\frac{f(z)}{z} \le \frac{1+z}{(1-z)(1-\alpha^2z^2)},$$

where $z \in U$, $\operatorname{Arg} 1 = 0$.

The last theorem is a generalization of the known result due to G.M. Golusin that concerns the estimation of $\operatorname{Arg}\frac{f(z)}{z}$ in the class T_R.

5. In view of Theorem 1.1 it is natural to consider some radius problem.

Definition 5.1. Let $\alpha, \beta \in \langle 0,1\rangle$. Denote by $R_\alpha(\beta)$ the radius of the largest disc $U_{R_\alpha(\beta)}$ such that every function f being an element of the class $\tilde{H}_\beta$ satisfies the following condition in the disc $U_{R_\alpha(\beta)}$:

$$\operatorname{Re}\left\{(1-\alpha^2z^2)\frac{f(z)}{z}\right\} > 0.$$

In other words, if $\alpha, \beta \in \langle 0,1\rangle$, then

$$R_\alpha(\beta) = \sup\left\{r \in \langle 0,1\rangle : \operatorname{Re}\left\{(1-\alpha^2z^2)\frac{f(z)}{z}\right\} > 0, \quad z \in U_r\right\}, \qquad f \in \tilde{H}_\beta.$$

We obtain

Theorem 5.1. *If $\alpha \in \langle 0,1\rangle$, then*

$$R_\alpha(0) = \left(\frac{2}{1+\alpha^2+(1+6\alpha^2+\alpha^4)^{1/2}}\right)^{1/2}.$$

From Theorem 5.1 we get immediately

Corollary 5.1. *Every function f of the class $\tilde{H}_0$ satisfies in the disc $U_{R_1(0)}$, where $R_1(0) = (\sqrt{2}-1)^{1/2}$, the following condition*

$$\operatorname{Re}\left\{(1-z^2)\frac{f(z)}{z}\right\} > 0.$$

The problem of finding $R_\alpha(\beta)$ for arbitrary $\alpha, \beta \in \langle 0,1\rangle$ remains open.

The omitted justification and other properties of the classes $\tilde{H}_\alpha$ and H_α wese submitted for publications ([4]).

References

1. G.M. Goluzin, *On typically real functions*, Mat. Sbornik (N.S.) **27** (1950), no. 69, 201–218. (in Russian)
2. W. Hengartner, G. Schober, *A remark on level curves for domains convex in one direction*, Applicable Analysis **3** (1973), 101–106.
3. G. Herglotz, *Über Potenzreihen mit positivem, reellem Teil im Einheitskreis*, Ber. Verh. Wis. Leipzig (1911), 501–511.

4. Z.J. Jakubowski, A. Lecko, *On certain classes of quasi-typically-real functions*, Demonstratio Mathematica (to appear).
5. M.S. Robertson, *On the coefficients of a typically-real functions*, Bull. Amer. Math. Soc. **41** (1935), 565–572.
6. W.W. Rogosinski, *Über positive harmonische Entwicklungen und typischreelle Potenzreihen*, Mat. Z. **35** (1932), 93–121.

O pewnych klasach funkcji quasi-typowo-rzeczywistych

Streszczenie. W pracy wprowadzono i zbadano klasy $\tilde{H}_\alpha$, $\alpha \in \langle 0,1\rangle$, funkcji $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$ holomorficznych w kole $U = \{z \in \mathbb{C} : |z| < 1\}$ i spełniających warunek $\operatorname{Re}\{(1-\alpha^2z^2)f(z)/z\} > 0$, $z \in U$. Rozważane są także podklasy $H_\alpha \subset \tilde{H}_\alpha$ funkcji o rzeczywistych współczynnikach ($a_n = \bar{a}_n$, $n = 2,3,\ldots$). Oczywiście klasa H_1 jest identyczna ze znaną rodziną T_R funkcji typowo-rzeczywistych w U.

Bronisławów, 11–15 stycznia, 1993 r.