MATERIAŁY NA XXXII KONFERENCJĘ SZKOLENIOWĄ Z GEOMETRII ANALITYCZNEJ I ALGEBRAICZNEJ ZESPOLONEJ

2011 **Łódź** **str. 11**

SUBALGEBRAS OF POLYNOMIAL ALGEBRAS CONTAINING PRIME POWERS OF VARIABLES

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Abstract

Let k be a field of arbitrary characteristic, let $k[x_1, \ldots, x_n]$ be the polynomial k-algebra and let p be a prime number. We describe subalgebras of the form $k[f_1, \ldots, f_n]$, where $f_1, \ldots, f_n$ are homogeneous polynomials, such that $k[x_1^p, \ldots, x_n^p] \subseteq k[f_1, \ldots, f_n]$.

Introduction

Throughout this paper n is a positive integer, p is a prime number and k is a field of arbitrary characteristic. By $k[x_1, \ldots, x_n]$ we denote the polynomial k-algebra in n indeterminates and by $\langle v_1, \ldots, v_m \rangle$ we denote the k-linear space spanned by the elements $v_1, \ldots, v_m$.

The following theorem was proved by Ganong in [3] under the additional assumption that the field k is algebraically closed, and without this assumption by Daigle in [2].

Theorem (Ganong, Daigle). *If k is a field of characteristic $p > 0$, A and B are polynomial k-algebras in two indeterminates such that $A^p \subsetneqq B \subsetneqq A$, where $A^p = \{a^p;\ a \in A\}$, then there exist $x, y \in A$ such that $A = k[x, y]$ and $B = k[x^p, y]$.*

It is natural to consider this problem in the case of n variables. In this paper we consider the homogeneous version of such a general problem, partially based on [6] and [7]. Note that some questions related to this problem were discussed in [4].

Denote by $T(n,p)$ the following statement:

"For every field k and arbitrary homogeneous polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$, such that

$$k[x_1^p, \ldots, x_n^p] \subseteq k[f_1, \ldots, f_n], \tag{1}$$

the following holds:

$$k[f_1, \ldots, f_n] = \begin{cases} k[x_1^{l_1}, \ldots, x_n^{l_n}], & \text{if } \operatorname{char} k \neq p, \\ k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n], & \text{if } \operatorname{char} k = p, \end{cases} \tag{2}$$

for some $l_1, \ldots, l_n \in \{1, p\}$, some $m \in \{0, 1, \ldots, n\}$, and some k-linear basis $y_1, \ldots, y_n$ of $\langle x_1, \ldots, x_n \rangle$."

The statement $T(n,p)$ was proved in [6] in the following particular cases:

1° $p = 2$ and arbitrary n,

2° $p = 3$ and arbitrary n,

3° $n = 2$ and arbitrary p,

4° $n = 3$ and $p \leqslant 19$,

5° $n = 4$ and $p \leqslant 7$.

In this paper we prove the statement $T(n,p)$ for arbitrary n and p (Theorem 2.2). In the proof we use a general form of well known Krull Theorem about principal ideals ([1], 5.2.10).

Theorem (Krull). *Let R be a noetherian commutative ring with unity, let P be a minimal prime ideal of an ideal generated by n elements. Then the height of P is not greater than n.*

Note a close connection of our problem with the question about polynomiality of the ring of constants of a derivation. If k is a field of characteristic $p > 0$, then the ring of constants of every k-derivation of $k[x_1, \ldots, x_n]$ contains $k[x_1^p, \ldots, x_n^p]$. Some information about rings of constants in positive characteristic can be found, for example, in [9], [10]. The ring of constants of a homogeneous k-derivation is always generated by homogeneous polynomials. Thus, the positive characteristic case of our problem is related to a description of rings of constants of homogeneous derivations, which are polynomial subalgebras (Theorem 3.1). The second author in [8] characterized linear derivations with trivial rings of constants and trivial fields of constants, in the case of $\operatorname{char} k = 0$ (see also [9]). The first author in

[5] obtained a description of linear derivations with ring of constants generated by linear forms. Now we know that, in the positive characteristic case, these are all rings of constants of linear derivations, which are polynomial k-algebras.

At the end of the paper (Theorem 3.3) we present a generalization of $T(n,p)$ for n arbitrary prime numbers, obtained in [7].

1 Preliminaries

Let k be a field, let n be a positive integer and let p be a prime number. Let $f_1,\dots,f_n \in k[x_1,\dots,x_n]$ be homogeneous polynomials of degrees $r_1,\dots,r_n$, respectively, satisfying (1). The following facts have been proved in [6]. We adopt their proofs to make our exposition complete.

Lemma 1.1 ([6], 2.3 a) *The polynomials $f_1,\dots,f_n$ are algebraically independent over k.*

Proof. By the assumption (1) we have the following field extensions

$$k \subseteq k(x_1^p,\dots,x_n^p) \subseteq k(f_1,\dots,f_n) \subseteq k(x_1,\dots,x_n).$$

We see that the transcendence degree of the field extension $k \subseteq k(f_1,\dots,f_n)$ is equal to n, so $f_1,\dots,f_n$ are algebraically independent over k. $\square$

Corollary 1.2 *The polynomials $f_1,\dots,f_n$ are nonzero.*

Lemma 1.3 ([6], 2.3 b) *For $i=1,\dots,n$ we can present x_i^p in the following form:*

$$x_i^p = \sum_{\substack{\alpha_1,\dots,\alpha_n \geqslant 0 \\ \alpha_1 r_1 + \dots + \alpha_n r_n = p}} a_\alpha^{(i)} f_1^{\alpha_1} \dots f_n^{\alpha_n}, \tag{3}$$

where $a_\alpha^{(i)} \in k$ for $\alpha = (\alpha_1,\dots,\alpha_n)$. The elements $a_\alpha^{(i)}$ are uniquely determined.

Proof. For $i=1,\dots,n$ the polynomial x_i^p belongs to $k[f_1,\dots,f_n]$ by the assumption (1). Then $x_i^p = F_i(f_1,\dots,f_n)$ for some polynomial $F_i \in k[T_1,\dots,T_n]$. Put

$$F_i = \sum_{\alpha_1,\dots,\alpha_n \geqslant 0} a_\alpha^{(i)} T_1^{\alpha_1} \dots T_n^{\alpha_n},$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$, $a_\alpha^{(i)} \in k$, and $a_\alpha^{(i)} \neq 0$ for only finitely many α. We obtain

$$x_i^p = \sum_{\alpha_1,\dots,\alpha_n \geqslant 0} a_\alpha^{(i)} f_1^{\alpha_1} \dots f_n^{\alpha_n}.$$

Recall that the polynomials $f_1, \dots, f_n$ are homogeneous of degrees $r_1, \dots, r_n$. Hence, each polynomial $f_1^{\alpha_1} \dots f_n^{\alpha_n}$ is homogeneous of degree $\alpha_1 r_1 + \dots + \alpha_n r_n$, and

x_i^p equals to the sum of all summands of degree p, that is, satisfying the equality $\alpha_1 r_1 + \ldots + \alpha_n r_n = p$.

Now, suppose that

$$\sum_{\substack{\alpha_1,\ldots,\alpha_n \geqslant 0 \\ \alpha_1 r_1 + \ldots + \alpha_n r_n = p}} a_\alpha^{(i)} f_1^{\alpha_1} \ldots f_n^{\alpha_n} = \sum_{\substack{\alpha_1,\ldots,\alpha_n \geqslant 0 \\ \alpha_1 r_1 + \ldots + \alpha_n r_n = p}} b_\alpha^{(i)} f_1^{\alpha_1} \ldots f_n^{\alpha_n}$$

for some $a_\alpha^{(i)}, b_\alpha^{(i)} \in k$. The polynomials $f_1, \ldots, f_n$ are algebraically independent over k (Lemma 1.1), so $a_\alpha^{(i)} = b_\alpha^{(i)}$ for every α. Thus the presentation (3) is unique. □

Lemma 1.4 ([6], 2.3 d) *The degrees $r_1, \ldots, r_n$ satisfy the inequalities*

$$1 \leqslant r_1, \ldots, r_n \leqslant p.$$

Proof. If $r_j = 0$ for some $j \in \{1, \ldots, n\}$, then $f_j \in k$, but it is impossible, because $f_1, \ldots, f_n$ are algebraically independent over k (Lemma 1.1).

If $r_j > p$ for some $j \in \{1, \ldots, n\}$, then in (3), in each equality

$$\alpha_1 r_1 + \ldots + \alpha_j r_j + \ldots + \alpha_n r_n = p$$

for $\alpha_1, \ldots, \alpha_n \geqslant 0$, we have $\alpha_j = 0$. Then from Lemma 1.3 for $i = 1, \ldots, n$ we have

$$x_i^p = \sum_{\substack{\alpha_1,\ldots,\alpha_n \geqslant 0 \\ \alpha_1 r_1 + \ldots + \alpha_n r_n = p}} a_\alpha^{(i)} f_1^{\alpha_1} \ldots f_{j-1}^{\alpha_{j-1}} f_{j+1}^{\alpha_{j+1}} \ldots f_n^{\alpha_n},$$

so $x_i^p \in k[f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n]$.

Hence

$$k[x_1^p, \ldots, x_n^p] \subseteq k[f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n],$$

and we obtain a field extension

$$k(x_1^p, \ldots, x_n^p) \subseteq k(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n),$$

where $\operatorname{tr\,deg}_k k(x_1^p, \ldots, x_n^p) = n$, $\operatorname{tr\,deg}_k k(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n) = n-1$; so we have a contradiction.

Therefore, for each $j \in \{1, \ldots, n\}$ we have $r_j > 0$ and $r_j \leqslant p$. □

The following proposition from [7] will be useful in the proof of Theorem 2.2.

Proposition 1.5 ([7]) *Let k be a field. Let $g_1, \ldots, g_s \in k[x_1, \ldots, x_n]$ be homogeneous polynomials, algebraically independent over k, of degrees $r_1 \leqslant \ldots \leqslant r_s$, respectively. Let $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$ be homogeneous polynomials of degrees $t_1 \leqslant \ldots \leqslant t_s$, respectively. Assume that*

$$k[g_1, \ldots, g_s] = k[h_1, \ldots, h_s].$$

Then $r_i = t_i$ for $i = 1, \ldots, s$.

Proof. Note that the polynomials $h_1, \ldots, h_s$ are also algebraically independent over k, because

$$\operatorname{tr\,deg}_k k(h_1, \ldots, h_s) = \operatorname{tr\,deg}_k k(g_1, \ldots, g_s) = s.$$

Similarly as in Lemma 1.3, for $i \in \{1, \ldots, s\}$, since $g_i \in k[h_1, \ldots, h_s]$, the following equation holds:

$$g_i = \sum_{\substack{\alpha_1, \ldots, \alpha_s \geqslant 0 \\ \alpha_1 t_1 + \ldots + \alpha_s t_s = r_i}} a_\alpha^{(i)} h_1^{\alpha_1} \ldots h_s^{\alpha_s},$$

where $a_\alpha^{(i)} \in k$ for $\alpha = (\alpha_1, \ldots, \alpha_s)$.

Suppose that $r_j < t_j$, where $j \in \{1, \ldots, s\}$. Then for $i \in \{1, \ldots, j\}$ and $l \in \{j, \ldots, s\}$ we have $r_i < t_l$, so $\alpha_l = 0$ in each equality

$$\alpha_1 t_1 + \ldots + \alpha_l t_l + \ldots + \alpha_s t_s = r_i.$$

If $j = 1$, then $i = 1$ and $\alpha_1 = \ldots = \alpha_s = 0$; a contradiction. If $j > 1$, then we obtain

$$g_i = \sum_{\substack{\alpha_1, \ldots, \alpha_{j-1} \geqslant 0 \\ \alpha_1 t_1 + \ldots + \alpha_{j-1} t_{j-1} = r_i}} b_\alpha^{(i)} h_1^{\alpha_1} \ldots h_{j-1}^{\alpha_{j-1}},$$

where $b_\alpha^{(i)} \in k$ for $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$. Therefore, the polynomials $g_1, \ldots, g_j$ belong to $k[h_1, \ldots, h_{j-1}]$, and we have a contradiction with transcendence degrees, as in the proof of Lemma 1.4. □

2 The main theorem

Recall that $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ are homogeneous polynomials of degrees $r_1, \ldots, r_n$, respectively, satisfying (1). It was proved in [6] that if these degrees are already equal 1 or p, the thesis of $T(n, p)$ holds.

Proposition 2.1 ([6], 2.5) *If $r_1, \ldots, r_n \in \{1, p\}$, then (2) holds.*

Proof. Assume that $r_1, \ldots, r_n = 1$. We know (Lemma 1.1) that $f_1, \ldots, f_n$ are algebraically independent over k, so $f_1, \ldots, f_n$ are linearly independent over k. Then $f_1, \ldots, f_n$ form a basis of the k-linear space $\langle x_1, \ldots, x_n \rangle$, and we have

$$k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n],$$

so (2) holds.

Now, assume that $r_1, \ldots, r_n = p$. Then the equality $\alpha_1 r_1 + \ldots + \alpha_n r_n = p$ from Lemma 1.3 is equivalent to $\alpha_1 + \ldots + \alpha_n = 1$. This means that $\alpha_j = 1$ for some j and $\alpha_l = 0$ for $l \neq j$, and the only summands in (3) are the following:

$$x_i^p = a_1^{(i)} f_1 + \ldots + a_n^{(i)} f_n,$$

where $a_j^{(i)} \in k$, for $i, j = 1, \ldots, n$. Hence $x_1^p, \ldots, x_n^p$ belong to the k-linear space $\langle f_1, \ldots, f_n \rangle$ and form a basis of this space. Therefore

$$k[f_1, \ldots, f_n] = k[x_1^p, \ldots, x_n^p],$$

and (2) holds.

Now, we may assume that

$$r_1 = \ldots = r_m = p,\ r_{m+1} = \ldots = r_n = 1$$

for some $m \in \{1, \ldots, n-1\}$, where $n \geqslant 2$. Note that $f_{m+1}, \ldots, f_n$ are linearly independent linear forms. We may choose elements $x_{j_1}, \ldots, x_{j_m}$, where $1 \leqslant j_1 < \ldots < j_m \leqslant n$, such that $x_{j_1}, \ldots, x_{j_m}, f_{m+1}, \ldots, f_n$ form a k-linear basis of $\langle x_1, \ldots, x_n \rangle$. We introduce new coordinates:

$$y_1 = x_{j_1}, \ldots, y_m = x_{j_m}, y_{m+1} = f_{m+1}, \ldots, y_n = f_n.$$

Consider a homomorphism of k-algebras $\varphi\colon k[y_1, \ldots, y_n] \to k[y_1, \ldots, y_m]$ such that $\varphi(y_i) = y_i$ for $i \leqslant m$ and $\varphi(y_i) = 0$ for $i > m$. Put $g_i = \varphi(f_i) \in k[y_1, \ldots, y_m]$ for $i \leqslant m$. Then

$$\varphi(k[f_1, \ldots, f_n]) = k[\varphi(f_1), \ldots, \varphi(f_m), \varphi(f_{m+1}), \ldots, \varphi(f_n)] = k[g_1, \ldots, g_m]$$

and

$$\varphi(k[x_1^p, \ldots, x_n^p]) = k[\varphi(x_1)^p, \ldots, \varphi(x_n)^p].$$

Note also that

$$k[y_1^p, \ldots, y_m^p] = k[\varphi(y_1)^p, \ldots, \varphi(y_m)^p] \subseteq k[\varphi(x_1)^p, \ldots, \varphi(x_n)^p].$$

Hence, applying the homomorphism φ to the inclusion (1), we obtain that

$$k[y_1^p, \ldots, y_m^p] \subseteq k[\varphi(x_1)^p, \ldots, \varphi(x_n)^p] \subseteq k[g_1, \ldots, g_m].$$

The polynomials $g_1, \ldots, g_m \in k[y_1, \ldots, y_m]$ are homogeneous of degree p, so in this case, as we have already observed above, $y_1^p, \ldots, y_m^p$ form a basis of the k-linear space $\langle g_1, \ldots, g_m \rangle$.

Let $j \in \{1, \ldots, m\}$. We have $g_j \in \langle y_1^p, \ldots, y_m^p \rangle$, so g_j is a linear combination of $x_{j_1}^p, \ldots, x_{j_m}^p$. By Lemma 1.3 we obtain

$$g_j = c_1^{(j)} f_1 + \ldots + c_m^{(j)} f_m + h_j,$$

where $c_1^{(j)}, \ldots, c_m^{(j)} \in k$, $h_j \in k[f_{m+1}, \ldots, f_n]$ and h_j is a homogeneous polynomial of degree p. Since $g_j \in \langle y_1^p, \ldots, y_m^p \rangle$, we have also

$$g_j = \varphi(g_j) = \varphi(c_1^{(j)} f_1 + \ldots + c_m^{(j)} f_m + h_j) = c_1^{(j)} g_1 + \ldots + c_m^{(j)} g_m.$$

This implies that $c_j^{(j)} = 1$ and $c_l^{(j)} = 0$ for $l \neq j$, because the polynomials $g_1, \ldots, g_m$ are linearly independent over k. Finally, $g_j = f_j + h_j$ for $j = 1, \ldots, m$, so

$$k[f_1, \ldots, f_n] = k[g_1, \ldots, g_m, f_{m+1}, \ldots, f_n] = k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n], \tag{4}$$

and we are done if $\operatorname{char} k = p$.

Assume that $\operatorname{char} k \neq p$. Put

$$\{1, \ldots, n\} \setminus \{j_1, \ldots, j_m\} = \{j_{m+1}, \ldots, j_n\},$$

where $1 \leqslant j_{m+1} < \ldots < j_n \leqslant n$. Take any $j \in \{j_{m+1}, \ldots, j_n\}$. Put $x_j = w_j + z_j$, where $w_j \in \langle y_1, \ldots, y_m \rangle$ and $z_j \in \langle y_{m+1}, \ldots, y_n \rangle$. Note that $z_j \neq 0$, because $x_j \notin \langle y_1, \ldots, y_m \rangle$. Consider $x_j^p = (w_j + z_j)^p$ as a polynomial in variables $y_1, \ldots, y_m$ over $k[y_{m+1}, \ldots, y_n]$. Observe that $pw_j z_j^{p-1}$ is the homogeneous component of degree 1 of this polynomial. On the other hand, we have $x_j^p \in k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n]$ by (1) and (4), so all nonzero homogeneous components (of x_j^p as a polynomial in variables $y_1, \ldots, y_m$ over $k[y_{m+1}, \ldots, y_n]$) have degrees divisible by p. Therefore $pw_j z_j^{p-1} = 0$, so $w_j = 0$ and $x_j \in \langle y_{m+1}, \ldots, y_n \rangle$. We obtain that the elements $x_{j_{m+1}}, \ldots, x_{j_n}$ belong to the k-linear space $\langle y_{m+1}, \ldots, y_n \rangle$, so they form a basis of this space. Finally, $\langle y_{m+1}, \ldots, y_n \rangle = \langle x_{j_{s+1}}, \ldots, x_{j_n} \rangle$, and

$$k[f_1, \ldots, f_n] = k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n] = k[x_{j_1}^p, \ldots, x_{j_m}^p, x_{j_{m+1}}, \ldots, x_{j_n}]. \ \square$$

Now we prove that $T(n, p)$ holds for arbitrary n and p.

Theorem 2.2 *Let n be a positive integer and let p be a prime number. For every field k of arbitrary characteristic and arbitrary homogeneous polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$, such that*

$$k[x_1^p, \ldots, x_n^p] \subseteq k[f_1, \ldots, f_n], \tag{1}$$

the following holds:

$$k[f_1, \ldots, f_n] = \begin{cases} k[x_1^{l_1}, \ldots, x_n^{l_n}], & \text{if } \operatorname{char} k \neq p, \\ k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n], & \text{if } \operatorname{char} k = p, \end{cases} \tag{2}$$

for some $l_1, \ldots, l_n \in \{1, p\}$, some $m \in \{0, 1, \ldots, n\}$, and some k-linear basis $y_1, \ldots, y_n$ of $\langle x_1, \ldots, x_n \rangle$.

Proof. Observe that $T(1, p)$ holds. Namely, if $k[x^p] \subseteq k[f]$ for some homogeneous polynomial $f \in k[x]$ of degree r, then $x^p = af^s$ for some $a \in k$ and $s \geqslant 1$ such that $rs = p$. Hence $r = 1$ or $s = 1$. If $r = 1$, then $k[f] = k[x]$. If $s = 1$, then $k[f] = k[x^p]$.

Let $n \geqslant 2$. Assume that $T(n-1, p)$ holds.

First, we will show that if $r_j = 1$ for some j, then (2) holds. If f_j is a linear form, then there exist $i \in \{1, \dots, n\}$ such that

$$x_1, \dots, x_{i-1}, f_j, x_{i+1}, \dots, x_n$$

form a k-linear basis of $\langle x_1, \dots, x_n \rangle$. Consider a homomorphism of k-algebras

$$\psi \colon k[x_1, \dots, x_n] \to k[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$$

such that $\psi(x_l) = x_l$ for each $l \neq i$ and $\psi(f_j) = 0$. Applying this homomorphism to the inclusion (1) we obtain that

$$k[x_1^p, \dots, x_{i-1}^p, x_{i+1}^p, \dots, x_n^p] \subseteq k[g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_n],$$

for some homogeneous polynomials $g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_n \in k[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ of degrees $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_n$, respectively.

By $T(n-1, p)$ we obtain that

$$k[g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_n] = \begin{cases} k[x_1^{l_1}, \dots, x_{i-1}^{l_{i-1}}, x_{i+1}^{l_{i+1}}, \dots, x_n^{l_n}], & \text{if } \operatorname{char} k \neq p, \\ k[z_1^p, \dots, z_{m'}^p, z_{m'+1}, \dots, z_{n-1}], & \text{if } \operatorname{char} k = p, \end{cases}$$

for some $l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_n \in \{1, p\}$, resp. for some $m' \in \{0, 1, \dots, n-1\}$ and some k-linear basis $z_1, \dots, z_{n-1}$ of $\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle$. In each case, $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_n \in \{1, p\}$ by Proposition 1.5. Hence (2) holds by Proposition 2.1.

Now we may assume that $r_1, \dots, r_n > 1$.

Recall (Lemma 1.4) that $r_1, \dots, r_n \leqslant p$. Suppose that $r_j < p$ for some $j \in \{1, \dots, n\}$, for example, $r_n < p$. For $i = 1, \dots, n$ we can present x_i^p in the following way:

$$x_i^p = g_1^{(i)} f_1 + \ldots + g_{n-2}^{(i)} f_{n-2} + h^{(i)},$$

where $g_j^{(i)} \in k[f_j, \dots, f_n]$ for $j = 1, \dots, n-2$ and $h^{(i)} \in k[f_{n-1}, f_n]$. More precisely (see Lemma 1.3):

$$h^{(i)} = \sum_{\substack{\alpha, \beta \geqslant 0 \\ \alpha r_{n-1} + \beta r_n = p}} a_{\alpha,\beta}^{(i)} f_{n-1}^{\alpha} f_n^{\beta}.$$

Since $1 < r_n < p$, we have always $\beta r_n \neq p$, so each equality $\alpha r_{n-1} + \beta r_n = p$ yields that $\alpha \neq 0$. This means that $h^{(i)}$ is divisible by f_{n-1}, and then x_i^p belongs to the ideal $I = (f_1, \dots, f_{n-1})$.

Let P be a minimal prime ideal of the ideal I. We have $x_i^p \in P$, so $x_i \in P$ for every $i \in \{1, \dots, n\}$. Thus $P = (x_1, \dots, x_n)$, because $(x_1, \dots, x_n)$ is a maximal ideal. On the other hand, by Krull Theorem, since the ideal I is generated by $n-1$ elements, the height of the ideal P is not greater than $n-1$; so we have a contradiction.

Hence, $r_j = p$ for each j, and (2) follows from Proposition 2.1. $\square$

Note that Theorem 2.2 gives a positive answer to Question I stated in [4]. We also obtain a partially positive answer to Question III, if we restrict it to the homogeneous case.

3 Some related problems

Recall that a k-linear map $d\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ such that $d(fg) = d(f)g + fd(g)$ for every $f, g \in k[x_1, \ldots, x_n]$, is called a *$k$-derivation* of $k[x_1, \ldots, x_n]$. Such a derivation is uniquely determined by polynomials $g_1 = d(x_1)$, $\ldots$, $g_n = d(x_n)$ and it is of the form

$$d = g_1 \cdot \frac{\partial}{\partial x_1} + \ldots + g_n \cdot \frac{\partial}{\partial x_n}. \tag{5}$$

The kernel of a k-derivation d is called the *ring of constants* and is denoted by $k[x_1, \ldots, x_n]^d$. If $\operatorname{char} k = p > 0$, then

$$k[x_1^p, \ldots, x_n^p] \subseteq k[x_1, \ldots, x_n]^d.$$

A derivation d of the above form (5) is called *homogeneous of degree s*, for some $s \in \mathbb{Z}$, if the polynomials g_1, $\ldots$, g_n are homogeneous of degree $s + 1$ (the zero polynomial is homogeneous of any degree). In this case, if $f \in k[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree r, then $d(f)$ is homogeneous of degree $r + s$. It is easy to observe that the ring of constants of a homogeneous derivation is a graded subalgebra. Therefore we can deduce the following from Theorem 2.2.

Theorem 3.1 *Let d be a homogeneous k-derivation of the polynomial algebra $k[x_1, \ldots, x_n]$, where k is a field of characteristic $p > 0$. Then $k[x_1, \ldots, x_n]^d$ is a polynomial k-algebra if and only if*

$$k[x_1, \ldots, x_n]^d = k[y_1^p, \ldots, y_m^p, y_{m+1}, \ldots, y_n]$$

for some $m \in \{0, 1, \ldots, n\}$ and some k-linear basis $y_1, \ldots, y_n$ of $\langle x_1, \ldots, x_n \rangle$.

A k-derivation d of $k[x_1, \ldots, x_n]$ is called *linear* if d is homogeneous of degree 0, that is, the polynomials g_1, $\ldots$, g_n in (5) are linear forms. The restriction of a linear derivation to the k-linear space $\langle x_1, \ldots, x_n \rangle$ is a k-linear endomorphism. Every endomorphism of $\langle x_1, \ldots, x_n \rangle$ uniquely determines linear forms g_1, $\ldots$, g_n, and then a unique linear k-derivation. We have the following corollary from the above theorem and Theorem 3.2 from [5].

Corollary 3.2 *Let d be a linear derivation of the polynomial algebra $k[x_1, \ldots, x_n]$, where k is a field of characteristic $p > 0$. Then $k[x_1, \ldots, x_n]^d$ is a polynomial*

k-algebra if and only if the Jordan matrix of the endomorphism $d|_{\langle x_1,\ldots,x_n\rangle}$ satisfies the following conditions.

(1) *Nonzero eigenvalues of different Jordan blocks are pairwise different and linearly independent over $\mathbb{F}_p$.*

(2) *At most one Jordan block has dimension greater than 1 and, if such a block exists, then:*

(*a*) *its dimension is equal to 2 in the case of $p > 2$,*

(*b*) *its dimension is equal to 2 or 3 in the case of $p = 2$.*

Now, let us note that Theorem 2.2 can be generalized in the following way.

Theorem 3.3 ([7]) *Let k be a field and let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ be homogeneous polynomials such that*

$$k[x_1^{p_1}, \ldots, x_n^{p_n}] \subseteq k[f_1, \ldots, f_n]$$

for some prime numbers $p_1, \ldots, p_n$.

a) *If $\operatorname{char} k \neq p_i$ for every $i \in \{1, \ldots, n\}$, then*

$$k[f_1, \ldots, f_n] = k[x_1^{l_1}, \ldots, x_n^{l_n}]$$

for some $l_1 \in \{1, p_1\}, \ldots, l_n \in \{1, p_n\}$.

b) *If $\operatorname{char} k$ belongs to the set $\{p_1, \ldots, p_n\}$, then*

$$k[f_1, \ldots, f_n] = k[y_1^{l_1}, \ldots, y_n^{l_n}]$$

for some $l_1 \in \{1, p_1\}, \ldots, l_n \in \{1, p_n\}$ and some k-linear basis $y_1, \ldots, y_n$ of $\langle x_1, \ldots, x_n\rangle$ such that

$$\begin{cases} \langle y_i;\ i \in T\rangle = \langle x_i;\ i \in T\rangle, \\ y_i = x_i \text{ for } i \in \{1, \ldots, n\} \setminus T, \end{cases}$$

where $T = \{i \in \{1, \ldots, n\};\ \operatorname{char} k = p_i\}$.

It may be interesting to ask, what can we say about subalgebras satisfying the following condition:

$$k[x_1^{m_1}, \ldots, x_n^{m_n}] \subseteq k[f_1, \ldots, f_n],$$

where $m_1, \ldots, m_n$ are positive integers.

Finally, note that in this article we have considered only homogenous cases. Ganong and Daigle solved the general (non-homogeneous) problem in two variables. A general problem, even for three variables, remains open.

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Łódź, 10 – 14 stycznia 2011 r.