Bi-Lipschitz equivalent cones with different degrees

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Łódź, January 9, 2024

In 1971, O. Zariski proposed many questions and the most known among them is the following.

Question A. Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, is it true that the multiplicities $m(V(f), 0)$ and $m(V(g), 0)$ are equal?

This is still an open problem. The stated version of Question A is Zariski's famous Multiplicity Conjecture. Recently, Zariski's Multiplicity Conjecture for families with isolated singularities was confirmed by Fernández de Bobadilla and Pełka.

We can consider the Zariski's Multiplicity Conjecture from the Lipschitz point of view:

General Metric Conjecture. Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$. If their germs at zero are bi-Lipschitz homeomorphic, then their multiplicities $m(X, 0)$ and $m(Y, 0)$ are equal.

Bobadilla, Fernandes and Sampaio proved that this Conjecture has a positive answer for $d = 2$. The positive answer for $d = 1$ was already known, since Neumann and Pichon, with previous contributions of Pham and Teissier and Fernandes, proved that the Puiseux pairs of plane curves are invariant under bi-Lipschitz homeomorphisms, and as a consequence the multiplicity of complex analytic curves with any codimension is invariant under bi-Lipschitz homeomorphisms.

However, in dimension three, Birbrair, Fernandes, Sampaio and Verbitsky have presented examples of complex algebraic cones X and Y with isolated singularity, which were bi-Lipschitz homeomorphic but with different multiplicities at the origin. Their proof was based on the theory of Smale-Barden manifolds.

The first aim of this Lecture is to generalize this result. We show that for every $k \geq 3$ there exist complex algebraic cones of dimension k with isolated singularities, which are bi-Lipschitz and semi-algebraically equivalent but have different degrees. Our proof is completely different than this of Birbrair, Fernandes, Sampaio and Verbitsky and it is based on the Steenrod Theorem about sphere bundles.

Let us recall:

Zariski Question B. (we give here somewhat simplified version) Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, is there a homeomorphism $h\colon E_0(V(f)) \to E_0(V(g))$?

Here $E_0(V(f))$ denotes the base of the cone tangent to $V(f)$ at 0.

This problem has a negative answer, as shown by Fernández de Bobadilla in 2005. However, in the bi-Lipschitz case it still makes sense and is still open:

> **Metric Question B.** Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions. If there is a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n,\ V(g), 0)$, is there a homeomorphism $h\colon E_0(V(f)) \to E_0\ (V(g))$?

The second question that we have in mind has the following more general statement:

> **General Metric Question B.** Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$. If $(X, 0)$ and $(Y, 0)$ are bi-Lipschitz homeomorphic, is there a homeomorphism $h : E_0(X) \to E_0(Y)$?

Kollár proved that if $X \subset \mathbb{CP}^{n+1}$ is a smooth projective hypersurface of dimension greater than one, then the degree of X is determined by the underlying topological space of X. Moreover Barthel and Dimca proved that in the case of projective hypersurfaces (possibly with singularities) of dimension greater than one, degree one is a topological invariant. Here we generalize these results to dimension $n > 2$. More precisely, we prove the following:

Theorem. Let $V, V' \subset \mathbb{CP}^{n+1}$ be two projective hypersurfaces. Assume $n > 2$. If V is homeomorphic to V', then $\deg V = \deg V'$.

As a consequence of this result, we show, that a positive answer to Metric Question B implies a positive answer to the following :

> **Metric Question A.** Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions. If there is a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, is it true that $m(V(f), 0) = m(V(g), 0)$?

In the final part of this lecture, we classify links of real cones with base $\mathbb{P}^1 \times \mathbb{P}^2$. As an application, we give examples of manifolds , which are not diffeomorphic to projective manifolds of odd degree.

Finally, we give an example of three four-dimensional real algebraic cones in $\mathbb{R}^8$ with isolated singularity which are semi-algebraically and bi-Lipschitz equivalent but have non-homeomorphic bases. In particular, the real version of General Metric Question B has a negative answer.

Definition

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two sets and let $h\colon X \to Y$.

1. We say that h is **Lipschitz** if there exists a positive constant C such that

$$\|h(x) - h(y)\| \leq C\|x - y\|, \quad \forall x, y \in X.$$

2. We say that h is **bi-Lipschitz** if h is a homeomorphism, it is Lipschitz and its inverse is also Lipschitz. In this case, we say that X and Y are **bi-Lipschitz equivalent**. When $n = m$ and h is the restriction of a bi-Lipschitz homeomorphism $H\colon \mathbb{R}^n \to \mathbb{R}^n$, we say that X and Y are **ambient bi-Lipschitz equivalent.**

Definition

Let $X \subset \mathbb{P}^n$ be an algebraic variety. Then by an algebraic cone $\overline{C(X)} \subset \mathbb{P}^{n+1}$ with base X we mean the set

$$\overline{C(X)} = \bigcup_{x \in X} \overline{O, x},$$

where O is the center of coordinates in $\mathbb{R}^{n+1}$, and $\overline{O, x}$ means the projective line which goes through O and x. By an affine cone $C(X)$ we mean $\overline{C(X)} \setminus X$.

Proposition

Let $C(X)$ and $C(Y)$ be affine cones in $\mathbb{R}^N$. Assume that their links are bi-Lipschitz (semi-algebraically) equivalent. Then they are bi-Lipschitz (semi-algebraically) equivalent. Moreover, if $\dim C(X) = \dim C(Y) = d$, $2d + 2 \leq N$ and $C(X)$ is semi-algebraically bi-Lipschitz equivalent to $C(Y)$, then they are ambient semi-algebraically bi-Lipschitz equivalent.

Theorem

Let C_k denotes the Veronese embedding of degree k of $\mathbb{CP}^1$ into $\mathbb{CP}^k$. Let $n \geq 2$ and consider the varieties $X_{k,n} = \phi(C_k \times \mathbb{CP}^{n-1})$, where ϕ is the Segre embedding. Then for fixed n all varieties $X_{k,n}$ have different degrees $\deg X_{k,n} = kn$ and among the cones $C(X_{k,n})$ there are infinitely many cones which are bi-Lipschitz and semi-algebraically equivalent.

Proof: Note that C_k is $k\mathbb{CP}^1$ as a cycle. Hence $C_k \times \mathbb{CP}^{n-1} \sim k\mathbb{CP}^1 \times \mathbb{CP}^{n-1}$. Since after the Segre embedding $\deg \mathbb{CP}^1 \times \mathbb{CP}^{n-1} = n$, we have $\deg X_{k,n} = kn$. Using a generic projection, we can assume that all $X_{k,n}$ are in $\mathbb{CP}^{2n+1}$. By construction, $X_{k,n}$ is the union of projective $(n-1)$-planes $X = \bigcup_{a \in C_k} \phi(\{a\} \times \mathbb{CP}^{n-1})$. This means that $\overline{C(X_{k,n})}$ is the union of n-planes which has the $(n-1)$-plane $\phi(\{a\} \times \mathbb{CP}^{n-1})$ at infinity and goes through the point $O = (0, ..., 0)$.

Thus the link $L_{k,n}$ of this cone is a union of $(2n-1)$-spheres. In fact using the Ehresmann Theorem, it is easy to observe that these links are sphere bundles over $C_k \cong S^2$ with projection being the a composition of a projection $p : \mathbb{CP}^{2n+1} \setminus \{0\} \to \mathbb{CP}^{2n}$ and the projection $q : C_k \times \mathbb{CP}^{n-1} \to C_k$. By the Steenrod Theorem topologically there are only two such sphere bundles.

On the other hand on a compact manifold of dimension different from four there are only a finite number of differential structures. This means that all manifolds $L_{k,n}$, $k = 1, 2, ...,$ can have only a finite number of different differential structures. By the Dirichlet box principle, among all $X_{k,n}$ there is an infinite family $\mathcal{S}$ whose members are diffeomorphic to each other.

By result of Kollar all links from the family $\mathcal{S}$ are Nash diffeomorphic. in particular they are bi-Lipschitz and semi-algebraically equivalent. Hence we see that all cones $C(X), X \in \mathcal{S}$, are bi-Lipschitz and semi-algebraically equivalent. But all members of the family $\{C(X) : X \in \mathcal{S}\}$ have different degrees.

Corollary

For every $n \geq 3$ *there exist two analytic* n *dimensional germs* $V, V' \subset (\mathbb{C}^{2n}, 0)$ *with isolated singularities, which are bi-Lipschitz, sub-analytically equivalent, but have different multiplicities at* 0.

Theorem

Let $V, V' \subset \mathbb{CP}^{n+1}$ be two projective hypersurfaces. Assume $n > 2$. If V is homeomorphic to V', then $\deg V = \deg V'$.

Proof: Let $V_1, \ldots, V_r$ (resp. $V'_1, \ldots, V'_s$) be the irreducible components of V (resp. V'). Let $\phi \colon V \to V'$ be a homeomorphism. We know that $\phi(V_j)$ is an irreducible component of V' for all $j = 1, ..., r$. Then $r = s$ and by reordering the indices if necessary, we can assume that $\phi(V_j) = V'_j$ for all $j = 1, ..., r$. Since $\deg V = \deg V_1 + ... + \deg V_r$ and $\deg V' = \deg V'_1 + ... + \deg V'_r$, we may assume that V and V' are irreducible projective hypersurfaces.

Let us recall that the cohomology ring of $\mathbb{CP}^{n+1}$ is isomorphic to $\mathbb{Z}[x]/(x^{n+2})$ and it is generated by the generator α of $H^2(\mathbb{CP}^{n+1},\mathbb{Z})$. Let $\iota : V \to \mathbb{CP}^{n+1}$ be the inclusion. By Lefschetz theorem and our assumption, $\iota^* : H^2(\mathbb{CP}^{n+1},\mathbb{Z}) \to H^2(V,\mathbb{Z})$ is an isomorphism. In particular, the element $\alpha_V = \iota^*(\alpha)$ is a generator of $H^2(V,\mathbb{Z})$.

Since we have a canonical epimorphism $H^{2n}(V,\mathbb{Z}) \to H_{2n}(V,\mathbb{Z})^*$ we see that these spaces are isomorphic. In fact, the mapping $H^{2n}(\mathbb{CP}^{n+1},\mathbb{Z}) \to H^{2n}(V,\mathbb{Z})$ is dual to the mapping $H_{2n}(V,\mathbb{Z}) \to H_{2n}(\mathbb{CP}^{n+1},\mathbb{Z})$. Since V as a topological cycle is equivalent to $\deg V \cdot H$, where H is a hyperplane (i.e. a generator of $H_{2n}(\mathbb{CP}^{n+1},\mathbb{Z})$) we see that the mapping $H_{2n}(V,\mathbb{Z}) \to H_{2n}(\mathbb{CP}^{n+1},\mathbb{Z})$ is multiplication by $\deg V$. Hence also the mapping $H^{2n}(\mathbb{CP}^{n+1},\mathbb{Z}) \to H^{2n}(V,\mathbb{Z})$ is multiplication by $\deg V$. This means that $\iota^*(\alpha^n) = \alpha_V^n = \deg V \cdot [V]^*$ where $[V]^*$ is the (dual) fundamental class.

Now let $\alpha_{V'}$ be a generator of $H^2(V', \mathbb{Z})$ constructed in an analogous way to α_V. Hence by symmetry we have $\alpha_{V'}^n = \deg V' \cdot [V']^*$. Let $\phi\colon V \to V'$ be a homeomorphism. Hence $\phi^*(\alpha_{V'}) = \pm\alpha_V$. Thus

$$\pm \deg V' \cdot [V]^* = \phi^*(\deg V' \cdot [V']^*) = \phi^*(\alpha_{V'}^n) = \pm\alpha_V^n = \pm \deg V \cdot [V]^*.$$

Hence $\deg V = \deg V'$.

Corollary

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two complex analytic functions with $n > 4$. Assume that there is a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$. If there is a homeomorphism $h\colon E_0(V(f)) \to E_0(V(g))$ then $m(V(f), 0) = m(V(g), 0)$.

Thus, we obtain the following:

Corollary If Metric Question B has a positive answer then Metric Question A has a positive answer as well.

In this section, we consider real algebraic varieties. We prove the following:

Theorem Let $\iota : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^n$ be an algebraic embedding. Let $X = \iota(\mathbb{P}^1 \times \mathbb{P}^2)$. If $\deg X$ is odd then the link of the cone $C(X)$ is diffeomorphic to the twisted product $S^1 \tilde{\times} S^2$. If $\deg X$ is even, then every connected link of $C(X)$ is diffeomorphic either to $\mathbb{P}^1 \times \mathbb{P}^2$ or to $\mathbb{P}^1 \times S^2$ and both cases are possible.

Proof: Denote by A_k, B_l the Veronese embedding of $\mathbb{P}^1$ and $\mathbb{P}^2$ of degree k and l respectively. Now let $\phi : A_k \times B_l \to \mathbb{P}^{N(k,l)}$ be a suitable Segre embedding and denote by $W_{k,l}$ the image $\phi(A_k \times B_l)$. As in the previous section, we see that deg $W_{k,l} = 3kl$. Let $X_{k,l} = C(W_{k,l})$ be the cone with base $W_{k,l}$. Additionally denote by $L_{k,l}$ the link of this cone.

By the constructions every base $W_{k,1}$ is the union of planes

$$X = \bigcup_{a \in A_k} \phi(\{a\} \times \mathbb{P}^2).$$

This means that $\overline{X_{k,1}}$ is the union of 3-planes which have the plane $\phi(\{a\} \times \mathbb{P}^2)$ at infinity and go through the point $O = (0, ..., 0)$. Similarly $\overline{X_{1,l}}$ is the union of planes which have the line $\phi_1(\mathbb{P}^1 \times \{a\})$ at infinity and go through the point $O = (0, ..., 0)$.

Thus the link of $X_{k,1}$ is a union of spheres and the link of $X_{1,l}$ is a union of circles. In fact, it is easy to observe that the former link is a sphere bundle over $\mathbb{P}^1$ whose projection is a composition of the projection $p : \mathbb{R}^8 \setminus \{0\} \to \mathbb{P}^7$ and the projection $q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$. Similarly the link of $X_{1,k}$ is a circle bundle over $\mathbb{P}^2$ whose projection is a composition of the projection $p : \mathbb{R}^8 \setminus \{0\} \to \mathbb{P}^7$ and the projection $q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$. In particular, both links are connected. Note that the link $L_{1,1}$ has the structure of a circle bundle over $\mathbb{P}^2$ and the structure of a sphere bundle over $\mathbb{P}^1$.

We have:

Lemma If the link over a cone with base $\mathbb{P}^1 \times \mathbb{P}^2$ is connected, then it is diffeomorphic either to $\mathbb{P}^1 \times \mathbb{P}^2$, or to $\mathbb{P}^1 \times S^2$, or to the twisted product $S^1 \tilde{\times} S^2 = S^1 \times S^2/G$, where G is the group generated by the involution
$g : S^1 \times S^2 \ni (x, p) \mapsto (-x, -p) \in S^1 \times S^2$.

Since the link $L_{k,1}$ is a sphere bundle over $\mathbb{P}^1$, we have an exact sequence

$$0 = \pi_1(S^2) \to \pi_1(L_{k,1}) \to \pi_1(\mathbb{P}^1) \to 0,$$

hence $\pi_1(L_{k,1}) = \pi_1(\mathbb{P}^1) = \mathbb{Z}$.
Thus the link $L_{1,1}$ has to be diffeomorphic either to the twisted product $S^1 \tilde{\times} S^2$ or to $\mathbb{P}^1 \times S^2$.

Using the theory of Seifert manifolds we exclude the second possibility. Indeed, the following lemma is true:

Lemma If M is an orientable, Seifert fibered space with orbit surface $\mathbb{P}^2$ and less than two exceptional fibers, then M is homeomorphic either to a lens space $L(4n, 2n-1)$, or to a Seifert space with orbit space S^2 and three exceptional fibers with two of them of index two, or to a connected sum of two copies of $\mathbb{P}^3$. All the relevant fundamental groups are finite except $\pi_1(\mathbb{P}^3 \# \mathbb{P}^3) = \mathbb{Z}/2 * \mathbb{Z}/2$.

In particular, we see that the space $\mathbb{P}^1 \times S^2$ with fundamental group $\mathbb{Z}$ cannot be the total space of a circle bundle over $\mathbb{P}^2$. Thus $L_{1,1}$ is diffeomorphic to $S^1 \tilde{\times} S^2$.

Now consider the link $L_{1,2}$. Since it is a circle bundle over $\mathbb{P}^2$ it can be diffeomorphic either to $\mathbb{P}^1 \times \mathbb{P}^2$ or to the twisted product $S^1 \tilde{\times} S^2$. If the second possibility holds then we can lift an analytic mapping $W_{1,1} \to W_{1,2}$ to an analytic mapping $L_{1,1} \to L_{1,2}$ which preserves the Hopf fibration. This means in the terminology of our paper from Compositio ("On the Fukui-Kurdyka-Paunescu Conjecture") that there is an a-invariant subanalytic bi-Lipschitz mapping from $X_{1,1}$ to $X_{1,2}$. But this mapping has a-invariant graph and again by our paper we have $\deg X_{1,1} = \deg X_{1,2} \mod 2$, a contradiction. Hence $L_{1,2} = \mathbb{P}^1 \times \mathbb{P}^2$.

Now consider the link $L_{2,1}$. Its fundamental group is $\mathbb{Z}$, hence it is diffeomorphic either to the twisted product $S^1 \tilde{\times} S^2$ or to $\mathbb{P}^1 \times S^2$. By the same argument as above the first possibility is excluded. Hence $L_{2,1} = \mathbb{P}^1 \times S^2$.

To finish our proof we need the following:

Lemma Let $C(X) \subset \mathbb{R}^n$ be an algebraic cone of dimension $d > 1$ with connected base X. If $\deg C(X)$ is odd, then the link of $C(X)$ is connected.

Let X be as in Theorem and assume deg $C(X)$ is odd. If the link L of $C(X)$ is not equal to the twisted product $S^1 \tilde{\times} S^2$, then either $L = L_{1,2}$ or $L = L_{2,1}$. Since the degrees of the cones $X_{1,2}$ and $X_{2,1}$ are even, arguing as above we get a contradiction. In the same way we can prove that if deg $C(X)$ is even, then the link L cannot be diffeomorphic to $L_{1,1} = S^1 \tilde{\times} S^2$.

Theorem

There exist three semi-algebraically and bi-Lipschitz equivalent algebraic cones $C(X), C(Y), C(Z) \subset \mathbb{R}^8$ with non-homeomorphic smooth algebraic bases. In fact, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$, $Y \cong \mathbb{P}^1 \times S^2$ and $Z \cong S^1 \tilde{\times} S^2$. In particular, the real version of General Metric Question B has a negative answer.

Corollary

There exist four-dimensional algebraic cones $C(X), C(Y) \subset \mathbb{R}^8$ and a semi-algebraic bi-Lipschitz homeomorphism $\phi : C(X) \to C(Y)$ which transforms every ray Ox into the ray $O\phi(x)$ isometrically, but there is no homeomorphism $C(X) \to C(Y)$ which transforms every generatrix onto a generatrix.

Let $C(X), C(Y)$ be as in Theorem. If such a homeomorphism $C(X) \to C(Y)$ exists, then it induces a homeomorphism $X \to Y$, a contradiction.

Theorem

(1) The manifolds $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$ are not diffeomorphic to projective varieties of odd degree.
(2) Let $\iota : \mathbb{P}^n \to \mathbb{P}^N$ be an algebraic embedding. If $\deg \iota(\mathbb{P}^n)$ is odd, then the link of $C(\iota(\mathbb{P}^n))$ is S^n, while if $\deg \iota(\mathbb{P}^n)$ is even, then the link of $C(\iota(\mathbb{P}^n))$ is disconnected.
(3) A simply connected real projective variety of positive dimension cannot have odd degree.

THANK YOU FOR THE ATTENTION!!!!!