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# BIFURCATION VALUES AND TRAJECTORIES OF GRADIENT FIELDS

Michał Klepczarek [1] (Łódź)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semialgebraic analytic function and $\lambda$ be a bifurcation value of $f$. We prove that there exists a trajectory $x : (\alpha, \beta) \to \mathbb{R}^n$ of the gradient field of $f$ such that $\lim_{t\to\alpha} f(x(t)) = \lambda$ or $\lim_{t\to\beta} f(x(t)) = \lambda$.

## Introduction

In the 1960s René Thom [Th1] gave conditions ensuring the local topological triviality of smooth mappings. It turns out that for every polynomial $f : \mathbb{C}^n \to \mathbb{C}$ there exists a finite subset $\Sigma \subset \mathbb{C}$ such that the function $f$ is a locally trivial fibration over $\mathbb{C} \setminus \Sigma$. The smallest such subset of $\mathbb{C}$ is called the set of bifurcation values of the function $f$. In the case of complex polynomials with isolated singularities at infinity, due to the works of Pham and Parusiński (see [Ph] and [Pa]), it is well known that the set of bifurcation values of $f$ consists of critical values of $f$ and regular values at which the Malgrange condition fails. Many mathematicians tried to characterize the bifurcation set in more general case introducing different conditions such as: quasi-tameness, Malgrange condition, M-tameness.

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Usually when we want to construct a trivialization of $f$ over a neighbourhood of a regular value $c$ we use the flow of $\nabla f$. Therefore, it is important to study the properties of possible trajectories of $\nabla f$. It is well known that any bounded trajectory $x$ of an analytic function $f$ has a limit point. Moreover, Thom conjectured that such a trajectory has a tangent at its limit point. This claim is known as the Gradient Conjecture and was solved by Kurdyka, Parusiński and Mostowski (see [KMP] and [KM] ). Using similar techniques a related theorem on the behaviour of the incisors at infinity was proved by Grandjean ([Gr]). He showed that if $x$ is a bounded trajectory of the gradient field of a semialgebraic function $f$ of class $C^2$, then there exists a limit $x(t)/\|x(t)\|$. In this work Grandjean also shows that if $f(x(t)) \to \lambda$ then $\lambda$ is the asymptotic critical value of the function $f$.

In this paper we investigate an opposite question in some way. We prove that for each bifurcation value $\lambda$ of a semialgebraic analytic function (i.e Nash function) $f : \mathbb{R}^n \to \mathbb{R}$ we can find a trajectory $x : (\alpha, \beta) \to \mathbb{R}^n$ such that

$$\lim_{t\to\alpha} f \circ x(t) = \lambda \quad \vee \quad \lim_{t\to\beta} f \circ x(t) = \lambda,$$

i.e. the set of bifurcation values of $f$ is contained in the set of values $\lambda$ satisfying the above condition. The examples show that this set is substantially smaller than the set of asymptotic critical values.

In the proof, we use the flow of $\nabla f$ and some properties of differential equations.

## 1. Preliminaries

Denote by $F : G \to \mathbb{R}^n$ a mapping defined on an open subset $G \subset \mathbb{R}^{n+1}$ and consider the following system of differential equations

$$x' = F(t, x) \tag{1}$$

where $x = (x_1, ..., x_n)$. Assuming that through each point $(\tau, \eta) \in G$ there passes exactly one integral solution $\gamma(\tau, \eta) : I(\tau, \eta) \to \mathbb{R}^n$ of (1) defined on an open interval $I(\tau, \eta)$, we can define a set

$$V = \{(\tau, \eta, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}; (\tau, \eta) \in G, t \in I(\tau, \eta)\}$$

and a mapping $\Phi : V \to \mathbb{R}^n$ by

$$\Phi(\tau, \eta, t) = \gamma(\tau, \eta)(t) \ \ (\tau, \eta, t) \in V.$$

The mapping $\Phi$ is called the general solution of system (1).

It is well known that the general solution of system (1) is of the same class as the mapping $F$ (see for example [Na]). Namely we have

**Theorem 1.** *class of solution If the mapping $F$ is of class $C^m$ ($C^\infty$, analitic) than the general solution of system (1) exists and is also of class $C^m$ ($C^\infty$, analitic).*

In this paper we consider a particular type of system (1) where $G = \mathbb{R} \times W$ for some open set $W \subset \mathbb{R}^n$ and $F(t, x) = \nabla f(x)$ for $(t, x) \in G$. Any integral solution of system

$$x' = \nabla f(x) \tag{2}$$

we call a trajectory of the field gradient field of f (field $\nabla f$ in short).

An autonomy of the system (2) allows arbitrary time movements.

**Proposition 2.** *time movement Let $f : W \to \mathbb{R}$ be a function of class $C^2$ and $\Phi : V \to W$ be a general solution of system (2). For any point $(t_1, \xi, t_2) \in V$ and any $t_0 \in \mathbb{R}$ there is $(t_1 - t_0, \xi, t_2 - t_0) \in V$ and*

$$\Phi(t_1, \xi, t_2) = \Phi(t_1 - t_0, \xi, t_2 - t_0).$$

*In particular,*

$$\Phi(t_1, \xi, t_2) = \Phi(0, \xi, t_2 - t_1) = \Phi(t_1 - t_2, \xi, 0). \tag{3}$$

**Proof.** Let $\gamma = \gamma(t_1, \xi) : (\alpha, \beta) \to W$ be the trajectory of the $\nabla f$ field such that $\gamma(t_1) = \xi$. Then a mapping $\gamma^* : (\alpha - t_0, \beta - t_0) \to W$ defined as

$$\gamma^*(t) = \gamma(t + t_0)$$

is the only trajectory that passes through $(t_1 - t_0, \xi)$. Indeed,

$$\gamma^*(t_1 - t_0) = \gamma(t_1 - t_0 + t_0) = \gamma(t_1) = \xi,$$
$$(\gamma^*)'(t) = \gamma'(t + t_0) = \nabla f(\gamma(t + t_0)) = \nabla f(\gamma^*(t)) \text{ for } t \in (\alpha - t_0, \beta - t_0).$$

Therefore,

$$\Phi(t_1, \xi, t_2) = \gamma(t_2) = \gamma(t_2 - t_0 + t_0) = \gamma(t_2 - t_0) = \Phi(t_1 - t_0, \xi, t_2 - t_0),$$

which completes the proof. □

## 2. Main result

Let $W \subset \mathbb{R}^n$, $U \subset \mathbb{R}$ be open sets. We say that the function $f : W \to U$ of class $C^\infty$ is a $C^\infty$ fibration over $U$ if there exists $y \in U$ and a mapping $\Psi_1 : W \to f^{-1}(y)$ such that the mapping

$$\Psi = (\Psi_1, f) : W \ni x \mapsto (\Psi_1(x), f(x)) \in f^{-1}(y) \times U$$

is a diffeomorphism of class $C^\infty$. The mapping $\Psi$ is called a trivialisation $f$ of class $C^\infty$ over $U$.

We say that $\lambda \in \mathbb{R}$ is a typical value of a function $f : W \to \mathbb{R}$ if $f$ is a $C^\infty$ fibration over some neighbourhood of $\lambda$. Any number $\lambda$ that is not a typical value of $f$ is called a bifurcation value of $f$. By $B(f)$ we denote the set of all bifurcation values of $f$.

It is well known that for semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$ of class $C^1$ we have $B(f) \subset K(f)$, where

$$K(f) = \{\lambda \in \mathbb{R} : \exists_{x_k \subset \mathbb{R}^n} f(x_k) \to \lambda \wedge (1 + \|x_k\|)\|\nabla f(x_k)\| \to 0\}$$

is the set of generalized critical values of $f$. Clearly $K(f) = K_0(f) \cup K_\infty(f)$, where

$$K_\infty(f) = \{\lambda \in \mathbb{R} : \exists_{x_k \subset \mathbb{R}^n} \|x_k\| \to \infty \wedge f(x_k) \to \lambda \wedge (1 + \|x_k\|)\|\nabla f(x_k)\| \to 0\}$$

is called the set of asymptotic critical values of $f$ and $K_0(f)$ is the set of critical values of $f$. In our case $f : \mathbb{R}^n \to \mathbb{R}$ is an analytic semialgebraic function and the set $K(f)$ is finite (see for example [KOS]). Moreover, values of $f$ along each trajectory of the gradient field converge to a certain critical value. More precisely,

**Theorem 3.** *konceladuja w K(f) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic semilgebraic function. For each trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of the gradient field we have*

$$\lim_{t\to\alpha}(f\circ\gamma)(t) \in K(f) \quad \wedge \quad \lim_{t\to\beta}(f\circ\gamma)(t) \in K(f).$$

If the set $\gamma_{|(\alpha,\delta]}$ is unbounded for some $\delta \in (\alpha, \beta)$ the proof of the first equation can be found in [Gr]. In the case where $\gamma_{|(\alpha,\delta]}$ is bounded we can use Łojasiewicz Theorem [Lo] to show that $\lim_{t\to\alpha} f(\gamma(t)) = f(x_1) \in K_0(f)$ for some $x_1 \in \mathbb{R}^n$.

Our aim is to prove the following theorem:

**Theorem 4.** *tw eng Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic semialgebraic function and $\lambda_0$ be a bifurcation value of $f$. There exists a trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of the field $\nabla f$ such that*

$$\lim_{t\to\alpha}(f\circ\gamma)(t) = \lambda_0 \quad \vee \quad \lim_{t\to\beta}(f\circ\gamma)(t) = \lambda_0.$$

Unfortunately the implication in the above theorem cannot be reversed. If we denote by $A(f)$ the set of all $\lambda$ for which there exists trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of the field $\nabla f$ such that $\lim_{t\to\alpha}(f\circ\gamma)(t) = \lambda_0$ or $\lim_{t\to\beta}(f\circ\gamma)(t) = \lambda_0$, then we have $B(f) \subsetneq A(f) \subsetneq K(f)$. We will illustrate this fact with examples.

**Example 1.** Let $f(x, y) = y^3, (x, y) \in \mathbb{R}^2$.

Obviously $B(f) = \emptyset$ and $K(f) = 0$ and the trajectories are of the form

$$\gamma^1_{C_1,C_2}(t) = (C_1, -\frac{1}{3t + C_2}) \qquad t \in (-\infty, -\frac{C_2}{3}),$$
$$\gamma^2_{C_1,C_2}(t) = (C_1, -\frac{1}{3t + C_2}) \qquad t \in (-\frac{C_2}{3}, \infty),$$
$$\gamma^3_{C_1}(t) = (C_1, 0) \qquad t \in (-\infty, \infty).$$

Consequently $\emptyset = B(f) \subsetneq A(f) = K(f) = \{0\}$.

**Example 2.** Let $f(x, y) = \frac{y}{1+x^2}, (x, y) \in \mathbb{R}^2$. Consider the system $(x', y') = \nabla f(x, y)$, i.e. the following system

$$x' = -\frac{2xy}{(1 + x^2)^2}\ , \qquad y' = \frac{1}{1 + x^2} \tag{4}$$

and let $\gamma = (\gamma_x, \gamma_y) : (\alpha, \beta) \to \mathbb{R}^2$ be a trajectory of field $\nabla f$.

If there exists $t_0 \in (\alpha, \beta)$ such that $\gamma_x(t_0) = 0$, then $\gamma(t) = (0, t)$ for $t \in \mathbb{R}$ and $\lim_{t\to\infty} f(\gamma(t)) = \infty$.

Now assume that $\gamma_x(t) \neq 0$ for $t \in (\alpha, \beta)$. In this case by dividing equations in (4) we get

$$ln|\gamma_x(t)| + \frac{1}{2}\gamma_x^2(t) = -\gamma_y^2(t) + C \tag{5}$$

for some constant $C \in \mathbb{R}$. From $K_0(f) = \emptyset$ we conclude $\lim_{t\to\beta} \|\gamma(t)\| = \infty$.

(a) If $\lim_{t\to\beta}|\gamma_x(t)| = \infty$ then (5) gives a contradiction.
(b) If $\lim_{t\to\beta}|\gamma_y(t)| = \infty$ then from (5) we have $\lim_{t\to\beta}\gamma_x(t) = 0$. Therefore $\lim_{t\to\beta} f(\gamma(t)) = \infty$.

Using the same argument, we get $\lim_{t\to\alpha} f(\gamma(t)) = -\infty$. Summing up, we have $\emptyset = B(f) = A(f) \subsetneq K(f) = \{0\}$.

## 3. Proof of Theorem 4

We will precede the proof of Theorem 4 by two lemmas and a proposition.

**Lemma 5.** *darboux ang Let $\lambda_0 \in \mathbb{R}$ and $U$ be an open interval such that $U \setminus \{\lambda_0\} \subset \mathbb{R} \setminus K(f)$. For any trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ satisfying*
(i) $f(\gamma(\alpha,\beta)) \cap U \neq \emptyset$
(ii) $\lim_{t\to\alpha} f\circ\gamma(t) \neq \lambda_0 \neq \lim_{t\to\beta} f\circ\gamma(t)$,
*the inclusion $U \subset f(\gamma(\alpha,\beta))$ holds.*

**Proof.** Supposing the contrary, that there exists $y_0 \in U$ such that
$$\forall_{t\in(\alpha,\beta)} \;\; f\circ\gamma(t) \neq y_0.$$
Consider any $y_1 \in f(\gamma(\alpha,\beta)) \cap U$. Assume that $y_1 < y_0$. From the Darboux property we have
$$\forall_{t\in(\alpha,\beta)} \;\; f\circ\gamma(t) < y_0.$$
Since $f\circ\gamma$ is a nondecreasing function (because $(f\circ\gamma)'(t) = \|\gamma'(t)\|^2 \geqslant 0$), so
$$\lim_{t\to\beta}(f\circ\gamma)(t) \in [y_1, y_0] \subset U \subset (\mathbb{R}\setminus K(f)) \cup \{\lambda_0\}.$$
From the assumption (ii) we get that $\lim_{t\to\beta}(f\circ\gamma)(t) \in \mathbb{R}\setminus K(f)$. On the other hand from Theorem 3 we have $\lim_{t\to\beta}(f\circ\gamma)(t) \in K(f)$ which gives a contradiction. In the case $y_1 > y_0$ consider $\lim_{t\to\alpha}(f\circ\gamma)(t)$ similarly as above. □

Suppose that $\lambda_0 \in f(\mathbb{R}^n)$ and $\nabla f(x) \neq 0$ for $x \in f^{-1}(\lambda_0)$. Take any $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap K(f) \subset \{\lambda_0\}$. Denote $U = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and let $\Phi : V \to f^{-1}(U)$ be the general solution of the system $x' = \nabla f(x)$, where $V = \{(\tau, \eta, t) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}; (\tau,\eta) \in \mathbb{R}\times f^{-1}(U), t \in I(\tau,\eta)\}$. Additionally, we will assume that each trajectory $\gamma : (\alpha,\beta) \to \mathbb{R}^n$ of the field $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies :

(ii) $$\lim_{t\to\alpha} f\circ\gamma(t) \neq \lambda_0 \neq \lim_{t\to\beta} f\circ\gamma(t).$$

For that specified $\lambda_0, U, V, \Phi$ we introduce the following indications.

For $x \in f^{-1}(U)$ we define $t_x$ as a real number for which $f\circ\Phi(0,x,t_x) = \lambda_0$. We show that

**Fact 1.** The number $t_x$ is well defined.

Indeed, suppose that there exists $x_0 \in f^{-1}(U)$ such that
$$\Phi(0, x_0, t) \notin f^{-1}(\lambda_0) \quad \text{for } t \in I(0, x_0).$$
Then there exists a trajectory $\gamma : (\alpha,\beta) \to \mathbb{R}^n$ satisfying (ii) and
$$f(\gamma(0)) = f(x_0) \in U \;\;\wedge\;\; \lambda_0 \notin f(\gamma(\alpha,\beta)),$$

which contradicts Lemma 5. The uniqueness of $t_x$ follows immediately from

$$(f \circ \gamma)'(t) = \|\nabla f(\gamma(t))\|^2 > 0, \quad t \in I(0, x_0),$$

as $\nabla f(x) \neq 0$ for $x \in f^{-1}(U)$. This gives the assertion of the Fact 1.

Now let $\xi \in f^{-1}(U)$ and $\mu \in U$. Denote by $t^\mu_\xi$ a real number for which $f \circ \Phi(t^\mu_\xi, \xi, 0) = \mu$. By using Property 2 ($\Phi(t^\mu_\xi, \xi, 0) = \Phi(0, \xi, -t^\mu_\xi)$) similarly as above we can show that:

**Fact 2.** The number $t^\mu_\xi$ is well defined.

The smoothness of $f$ implies the following

**Fact 3.** The functions

$$T : f^{-1}(U) \ni x \to t_x \in \mathbb{R},$$
$$T^* : f^{-1}(U) \times U \ni (\xi, \mu) \to t^\mu_\xi \in \mathbb{R}$$

are smooth.

Indeed, take any $x_0 \in f^{-1}(U)$. By definition, $t_x$ satisfies

$$(f \circ \Phi)(0, x_0, t_{x_0}) = \lambda_0$$

and the function $(f \circ \Phi)(0, \cdot, \cdot)$ is of class $C^\infty$ (see Theorem 1) such that

$$(f \circ \Phi)'_t(0, x_0, t_{x_0}) = (f \circ \gamma)'_t(t_{x_0}) = \|\nabla f(\gamma(t_{x_0})\|^2 > 0,$$

where $\gamma = \gamma(0, x_0)$. Thus, using the Implicit Function Theorem, there are neighbourhoods: $H$ of $x_0$ and $K$ of $t_{x_0}$ and a function $R : H \to K$ such that for every $x \in H$ the point $t = R(x)$ is the only solution of

$$(f \circ \Phi)(0, x, t) = \lambda_0$$

in $K$. Moreover, $R$ is of $C^\infty$ class. In consequence $R = T_{|H}$, so the function $T$ is smooth. The smothness of the function $T^*$ can be obtained analogously by considering the function $(\xi, \mu, t) \to (f \circ \Phi)(0, \xi, -t) - \mu$.

**Proposition 6.** *wl ang Let $\lambda_0 \in f(\mathbb{R}^n)$, $\nabla f(x) \neq 0$ for $x \in f^{-1}(\lambda_0)$ and let $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap K(f) \subset \{\lambda_0\}$ for some $\varepsilon > 0$. Denote $U = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. If every trajectory of $x' = \nabla f(x)$ satisfies* (ii)*, then*

(a) $\Phi(t_1, \Phi(0, x, t_1), 0) = x$ *for $x \in f^{-1}(U)$, $t_1 \in I(0, x)$*

(b) $t = t^{f(x)}_{\Phi(0,x,t)}$ *for $x \in f^{-1}(U)$, $t \in I(0, x)$.*

**Proof.** (a) Let $\gamma = \gamma(0, x) : I(0, x) \to f^{-1}(U)$. Take any $t_1 \in I(0, x)$ and denote $\xi = \gamma(t_1) = \Phi(0, x, t_1)$. Then

$$\Phi(t_1, \Phi(0, x, t_1), 0) = \Phi(t_1, \xi, 0) = \gamma(0) = x.$$

(b) Let $\xi = \Phi(0, x, t)$ and $\gamma = \gamma(0, x)$. Obviously, $\gamma(t) = \xi$. Therefore

$$(f \circ \Phi)(t, \xi, 0) = f(\gamma(0)) = f(x).$$

Moreover, from the definition of $t^\mu_\xi$ we have

$$(f \circ \Phi)(t^{f(x)}_\xi, \xi, 0) = f(x),$$

and taking into account Proposition 2 and the monotonicity of $f \circ \gamma$, we obtain $t = t_\xi^{f(x)} = t_{\Phi(0,x,t)}^{f(x)}$. □

**Lemma 7.** *trywializacja eng Under the assumptions of Proposition 6, $\lambda_0$ is a typical value of $f$.*

**Proof.** Let $V = \{(\tau, \eta, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}; (\tau, \eta) \in \mathbb{R} \times f^{-1}(U), t \in I(\tau, \eta)\}$ and $\Phi : V \to f^{-1}(U)$ be a general solution of $x' = \nabla f(x)$ (in $f^{-1}(U)$). Define a mapping

$$\Psi : f^{-1}(U) \ni x \to (\Phi(0, x, t_x), f(x)) \in f^{-1}(\lambda_0) \times U,$$
$$\Theta : f^{-1}(\lambda_0) \times U \ni (\xi, \mu) \to \Phi(t_\xi^\mu, \xi, 0) \in f^{-1}(U).$$

Clearly $\Psi$ and $\Theta$ are of class $C^\infty$. We will show that $\Psi = \Theta^{-1}$. Take any $x \in f^{-1}(U)$. Then

$$\Theta \circ \Psi(x) = \Theta(\Phi(0, x, t_x), f(x)) = \Phi(t_{\Phi(0,x,t_x)}^{f(x)}, \Phi(0, x, t_x), 0).$$

Using Proposition 6 we get

$$\Theta \circ \Psi(x) = \Phi(t_{\Phi(0,x,t_x)}^{f(x)}, \Phi(0, x, t_x), 0) = \Phi(t_x, \Phi(0, x, t_x), 0) = x.$$

Now consider any $(\xi, \mu) \in f^{-1}(\lambda_0) \times U$ and denote $\gamma = \gamma(0, \xi)$. From Lemma 5 there exists $t_0 \in I(0, \xi)$ such that $(f \circ \gamma)(t_0) = \mu$. Denote $x = \gamma(t_0)$. Then $\xi = \Phi(0, x, t_x)$ and $\mu = f(x)$. Using Proposition 6 we have $t_\xi^\mu = t_{\Phi(0,x,t_x)}^{f(x)} = t_x$ and

$$\Phi(t_\xi^\mu, \xi, 0) = \Phi(t_x, \Phi(0, x, t_x), 0) = x.$$

Therefore

$$(\Psi \circ \Theta)(\xi, \eta) = \Psi(\Phi(t_\xi^\mu, \xi, 0)) = \Psi(x) =$$
$$= (\Phi(0, x, t_x), f(x))) = (\xi, \mu).$$

Summarising, $\Psi = \Theta^{-1}$ and $\Psi$ is $C^k$ trivialisation of $f$ over $U$. □

Now we can proceed to the proof of Theorem 4.

**Proof of Theorem 3.** Firstly, let us consider the case when $\lambda \in K_0(f)$. Take any $x_0 \in \mathbb{R}^n$ such that $\nabla f(x_0) = 0$. Then $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(t) = x_0$ for $t \in \mathbb{R}$, is the trajectory of $\nabla f$ field satisfying $\lambda_0 = \lim_{t\to\infty} f \circ \gamma(t)$.

Now let $\lambda_0 \in (B(f) \setminus K_0(f)) \cap f(\mathbb{R}^n)$ and suppose that for each trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ there is

(ii) $$\lim_{t\to\alpha}(f \circ \gamma)(t) \neq \lambda_0 \neq \lim_{t\to\beta}(f \circ \gamma)(t).$$

The finiteness of $K(f)$ allows us to take an open interval $U$ such that $K(f)$ and $U \cap K(f) \subset \{\lambda_0\}$. Using Lemma 7 we get that $\lambda_0$ is a typical value of $f$, which is contrary to the assumptions.

Finally, let $\lambda_0 \in B(f) \setminus f(\mathbb{R}^n)$. Then $\lambda_0$ must belong to the closure of $f(\mathbb{R}^n)$ ($f_{|\emptyset}$ is a $C^\infty$ fibration). Take any open interval $U$ such that $U \cap K(f) \subset \{\lambda_0\}$ and $f(x_0) = y_0 \in U \cap f(\mathbb{R}^n)$. If $\lambda_0 = \sup\ f(\mathbb{R}^n)$ then using Theorem 3 for trajectory $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ such that $\gamma(0) = x_0$, we have $\lim_{t\to\beta}(f \circ \gamma)(t) \in U \cap K(f) \subset \{\lambda_0\}$, which proves the claim in this case. In the case of $\lambda_0 = \inf\ f(\mathbb{R}^n)$ we consider $\lim_{t\to\alpha}(f \circ \gamma)(t)$. □

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## WARTOŚCI BIFURKACYJNE I TRAJEKTORIE POLA GRADIENTOWEGO

Niech $f : \mathbb{R}^n \to \mathbb{R}$ będzie semialgebraiczną funkcją analityczną i niech $\lambda$ będzie wartością bifurkacyjną funkcji $f$. W pracy dowodzimy, że istnieje trajektoria $x : (\alpha, \beta) \to \mathbb{R}^n$ pola gradientowego funkcji $f$ taka, że $\lim_{t\to\alpha} f(x(t)) = \lambda$ lub $\lim_{t\to\beta} f(x(t)) = \lambda$.

Faculty of Mathematics and Computer Science,
University of Łódź
S. Banacha 22, 90-238 Łódź, Poland
E-mail: mikimiki88@tlen.pl