Blown-up Čech cohomology and Cartan's Theorem B in real algebraic geometry

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Theorem 1 (Serre)

Let X be an affine variety over an algebraically closed field and $\mathcal{F}$ be a coherent sheaf on X. Then $\check{H}^p(X, \mathcal{F}) = 0$ for $p > 0$.

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X will be a non-singular irreducible real affine variety with structure sheaf $\mathcal{O}_X$.

Definition 2

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called quasi-coherent if there exists a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X such that for every U_i there is an exact sequence of sheaves

$$\mathcal{O}_X^{\oplus J_i}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

$\mathcal{F}$ is called coherent if the sets I_i, J_i can be taken to be finite.

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Definition 3

We say that a regular function $g : X \to \mathbb{R}$ on a non-singular real algebraic variety of dimension d is a simple normal crossing if in a neighbourhood of each point $a \in X$ one has

$$g(x) = u(x)x^{\alpha} = u(x)x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d}$$

where $u(x)$ is a unit at a, $\alpha \in \mathbb{N}^d$ and $x = (x_1, x_2, \dots, x_d)$ are local coordinates near a, i.e. $x_1, x_2, \dots, x_d \in \mathcal{O}_{a,X}$ is a regular system of parameters of the local ring $\mathcal{O}_{a,X}$.

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Theorem 4

Let $f_1, f_2, \dots, f_k$ be regular functions on X. Then there exists a multi-blowup $\sigma : \widetilde{X} \to X$ such that $\sigma^* f_1, \sigma^* f_2, \dots, \sigma^* f_k$ are simple normal crossings and they are linearly ordered by divisibility relation near each point $b \in \widetilde{X}$.

Given two multi-blowups $\alpha : X_\alpha \to X,\ \beta : X_\beta \to X$ we say that $X_\alpha \succeq X_\beta$ if there is a (unique) regular map $f_{\alpha\beta}$ making the following diagram commute

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \\ & \searrow^{\alpha} & \downarrow{\scriptstyle\beta} \\ & & X \end{array}$$

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Proposition 1

With the order given above, the set of multi-blowups of X is a directed set.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X. Put $U_{i_0 \dots i_q} = U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_q}$ and

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{1 \le i_0 \le i_1 \le \dots \le i_q \le n} \mathcal{F}(U_{i_0 i_1 \dots i_q}).$$

$C^q(\mathcal{U}, \mathcal{F})$ is called the abelian group of q-cochains.

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We have a chain complex

$$\dots \longrightarrow C^{q-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{q-1}} C^q(\mathcal{U}, \mathcal{F}) \xrightarrow{d^q} C^{q+1}(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

where

$$(d^q f)_{i_0 i_1, \dots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 i_1 \dots \widehat{i_j} \dots i_{q+1}} |_{U_{i_0 i_1 \dots i_{q+1}}}$$

for any $f = (f_{i_0 \dots i_q}) \in C^q(\mathcal{U}, \mathcal{F})$.

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for any $f = (f_{i_0 \dots i_q}) \in C^q(\mathcal{U}, \mathcal{F})$. If $\alpha : X_\alpha \to X$ is a morphism of real affine varieties, we get the induced chain complex

$$\dots \longrightarrow C^{q-1}(\mathcal{U}^\alpha, \alpha^* \mathcal{F}) \xrightarrow{d^{q-1}} C^q(\mathcal{U}^\alpha, \alpha^* \mathcal{F}) \xrightarrow{d^q} C^{q+1}(\mathcal{U}^\alpha, \alpha^* \mathcal{F}) \longrightarrow \dots .$$

and a canonical chain complex homomorphism

$$\alpha^* : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}^\alpha, \alpha^* \mathcal{F});$$

Put

$$\widetilde{\mathcal{F}}(U) = \varinjlim_{\alpha} \alpha^* \mathcal{F}(U^\alpha),$$

for any Zariski open subset U of X; direct limit is taken over the directed set of multi-blowups of X. Obviously, $\widetilde{\mathcal{F}}(U)$ has a structure of $\mathcal{O}_X(U)$-module.

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Definition 5

The q-th blown-up Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ $\widetilde{H}^q(\mathcal{U}, \mathcal{F})$ is the q-th cohomology group of the chain complex $\widetilde{C}^\bullet(\mathcal{U}, \mathcal{F})$.

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What do we have to assume about $\mathcal{F}, \mathcal{G}, \mathcal{H}$ in order to obtain a short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\phi} \widetilde{C}^\bullet(\mathcal{U}, \mathcal{G}) \xrightarrow{\psi} \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}) \to 0?$$

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In general we have only

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{H})$$

and if $\alpha : X_\alpha \to X$ is a multi-blowup we have even less

$$C^\bullet(\mathcal{U}^\alpha, \alpha^*\mathcal{F}) \to C^\bullet(\mathcal{U}^\alpha, \alpha^*\mathcal{G}) \to C^\bullet(\mathcal{U}^\alpha, \alpha^*\mathcal{H}).$$

Lemma 6

Let $\mathcal{F}$ be a quasi-coherent sheaf on X. For any $Q \in \mathcal{O}_X(X)$ and a section $s \in \mathcal{F}(X)$ such that $s|_U = 0$ with $U = X \setminus \{Q = 0\}$, there exist a multi-blowup $\sigma : \widetilde{X} \to X$ and a positive integer N such that $(Q^N)^\sigma \sigma^* s = 0$ in $\sigma^* \mathcal{F}(\widetilde{X})$.

Lemma 7

Let $\mathcal{F}$ be a quasi-coherent sheaf on X with local presentations

$$\mathcal{O}_X^{\oplus J}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0 \quad i = 1, 2, \ldots, n$$

on a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X. Consider a finite number of sections $s_j \in \mathcal{F}(V_j)$ on Zariski open sets

$$V_j = X \setminus \{Q_j = 0\}, \quad j = 1, 2, \ldots, m$$

where Q_j are regular functions on X. Assume that every V_j is contained in $U_{i(j)}$ for some $i(j) = 1, 2, \ldots, n$ and that for each j there is a section $t_j \in \mathcal{O}_X^{I_{i(j)}}(V_j)$ such that $\psi_{i(j)}(t_j) = s_j$. Then there exists a positive integer N and a multi-blowup $\sigma : X_\sigma \to X$ such that every section $(Q_j^N)^\sigma \sigma^* s_j, \quad j = 1, 2, \ldots, m$, extends to a global section on X_σ.

Theorem 8

Let

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X. Then for any Zariski open $U \subset X$ and any section $u \in \mathcal{H}(U)$ there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^* u \in \operatorname{im} \theta^\alpha$.

Theorem 8

Let

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X*. Then for any Zariski open* $U \subset X$ *and any section* $u \in \mathcal{H}(U)$ *there exists a multi-blowup* $\alpha : X_\alpha \to X$ *such that* $\alpha^* u \in \operatorname{im} \theta^\alpha$.

Corollary 1

Let

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X*. Then, for any Zariski open subset* $U \subset X$*, the induced sequence of* $\mathcal{O}_X(U)$*-modules*

$$\widetilde{\mathcal{G}}(U) \to \widetilde{\mathcal{H}}(U) \to 0$$

is exact.

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Definition 9

We say that $\mathcal{H}$ is of homological dimension k at x, $\operatorname{hdim}_x \mathcal{H} = k$, if k is the smallest integer such that there exists a Zariski open neighbourhood U and sets of indices $I_0, I_1, \ldots, I_k$ for which there is an exact sequence of sheaves:

$$0 \to \mathcal{O}_X^{\oplus I_k}|_U \to \mathcal{O}_X^{\oplus I_{k-1}}|_U \to \cdots \to \mathcal{O}_X^{\oplus I_0}|_U \to \mathcal{H}|_U \to 0.$$

We define the homological dimension of $\mathcal{H}$ as

$$\operatorname{hdim} \mathcal{H} = \sup_{x \in X} \operatorname{hdim}_x \mathcal{H}.$$

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We define the homological dimension of $\mathcal{H}$ as

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Obviously, $\operatorname{hdim} \mathcal{H} = 0$ iff $\mathcal{H}$ is a locally free sheaf. Consequently, $\operatorname{hdim} \mathcal{H} = 1$ means that $\mathcal{H}$ is locally a quotient of free sheaves.

Proposition 2

Let $\mathcal{H}$ be a coherent sheaf on X and $x \in X$, then

$$\operatorname{pd} \mathcal{H}_x = \operatorname{hdim}{}_x \mathcal{H},$$

here $\operatorname{pd} \mathcal{H}_x$ is a projective dimension of $\mathcal{O}_{x,X}$-module.

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Proposition 3

Let R be an integral domain, and K its field of fractions. Consider an exact sequence of R-modules

$$0 \to F \to G$$

such that F is free and the projective dimension of G is ≤ 1. Then for any ring S such that $R \subset S \subset K$ the induced sequence

$$0 \to F \otimes_R S \to G \otimes_R S.$$

is exact.

Proposition 4

Let

$$0 \to \mathcal{F} \to \mathcal{G}$$

be an exact sequence of quasi-coherent sheaves on X *such that* $\operatorname{hdim} \mathcal{F} = 0$ *and* $\operatorname{hdim} \mathcal{G} \leq 1$*. Then for any multi-blowup* $\sigma : X_\sigma \to X$ *the induced sequence*

$$0 \to \sigma^* \mathcal{F} \to \sigma^* \mathcal{G}$$

is exact.

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is exact.

Lemma 10

Let $\alpha : X_\alpha \to X$ *be a multi-blowup of* X *and* $\mathcal{H}$ *a quasi-coherent sheaf on* X. *If* $\mathcal{H}$ *is of homological dimension* ≤ 1, *so is the pull-back* $\alpha^* \mathcal{H}$.

Proposition 5

Let $\mathcal{U}$ *be a finite Zariski open covering of* X *and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of quasi-coherent sheaves on X *such that* $\operatorname{hdim} \mathcal{F} = 0$ *and* $\operatorname{hdim} \mathcal{G} \leq 1$.

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Let $\mathcal{U}$ be a finite Zariski open covering of X and

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be a short exact sequence of quasi-coherent sheaves on X such that $\operatorname{hdim} \mathcal{F} = 0$ and $\operatorname{hdim} \mathcal{G} \leq 1$. Then there is an induced short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\phi} \widetilde{C}^\bullet(\mathcal{U}, \mathcal{G}) \xrightarrow{\psi} \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

which induces long exact sequence of blown-up Čech cohomology

$$\cdots \to \widetilde{H}^p(\mathcal{U}, \mathcal{F}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{G}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{p+1}(\mathcal{U}, \mathcal{F}) \to \dots$$

Proposition 6

Let $\mathcal{F}$ be a quasi-coherent locally free subsheaf of $\mathcal{O}_X^{\oplus l}$. For any finite Zariski open covering of X we have

$$\widetilde{H}^q(\mathcal{U},\mathcal{F}) = 0.$$

Proposition 6

Let $\mathcal{F}$ be a quasi-coherent locally free subsheaf of $\mathcal{O}_X^{\oplus l}$. For any finite Zariski open covering of X we have

$$\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = 0.$$

From this we immediately obtain Cartan's Theorem B

Theorem 11

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and let $\mathcal{U}$ be a finite Zariski open covering of X. Assume that one of the following conditions hold

a) *$\mathcal{F}$ is a quasi-coherent locally free subsheaf of $\mathcal{O}_X^{\oplus l}$.*

b) *$\mathcal{F}$ is a quasi-coherent sheaf with global presentation and $\operatorname{hdim} \mathcal{F} \leq 1$.*

c) *$\mathcal{F}$ is a coherent sheaf with $\operatorname{hdim} \mathcal{F} \leq 1$.*

Then $\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = 0$, for $q \geq 1$.

Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X. Assume that for each i we have a rational function f_i on U_i such that $f_i - f_j$ is regular on $U_i \cap U_j$ for each two distinct indices $i, j = 1, 2, \ldots n$. Then we call $\{(U_i, f_i)\}_{i=1}^n$ data of the first Cousin problem or an additive Cousin distribution on X.

Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X. Assume that for each i we have a rational function f_i on U_i such that $f_i - f_j$ is regular on $U_i \cap U_j$ for each two distinct indices $i, j = 1, 2, \dots n$. Then we call $\{(U_i, f_i)\}_{i=1}^n$ data of the first Cousin problem or an additive Cousin distribution on X.
The first Cousin problem consists in characterizing those data $\{(U_i, f_i)\}_{i=1}^n$ which have the principal parts of a rational function f on X, i.e. those for which $f - f_i$ are regular on U_i, $i = 1, 2, \dots, n$. We then say that the data $\{(U_i, f_i)\}_{i=1}^n$ is *solvable*. If every first Cousin data on X is solvable, we say that the first Cousin problem is *universally solvable on X*

Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \xrightarrow{\varphi} \mathcal{H}_X := \mathcal{K}_X/\mathcal{O}_X \to 0$$

of quasi-coherent sheaves on X. The data $\{(U_i, f_i)\}_{i=1}^n$ of the first Cousin problem can be related to a unique global section $s \in \mathcal{H}_X(X)$; every such section is called a principal part distribution on X. Then we also say that $\{(U_i, f_i)\}_{i=1}^n$ is an s-representing distribution. In particular, for every rational function $f \in \mathcal{K}_X(X)$ we have its principal part distribution $\varphi(f)$ on X.

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Clearly, every s-representing distribution $\{(U_i, f_i)\}_{i=1}^n$ determines a 1-cocycle $(g)_{ij}$, $g_{ij} = f_i - f_j$ which induces a cohomology class $\zeta(s)$ in $H^1(\mathcal{U}, \mathcal{O}_X)$. The solvability of a given Cousin data can be rephrased in terms of vanishing $\zeta(s)$ in $H^1(\mathcal{U}, \mathcal{O}_X)$.

Lemma 12

An s-representing distribution $\{(U_i, f_i)\}_{i=1}^n$ is solvable iff $\zeta(s) = 0 \in H^1(\mathcal{U}, \mathcal{O}_X)$.

Lemma 13

For any multi-blowup $\alpha : X_\alpha \to X$ *we have a short exact sequence*

$$0 \to \mathcal{O}_{X_\alpha} \to \mathcal{K}_{X_\alpha} \to \mathcal{H}_{X_\alpha} \to 0$$

of quasi-coherent sheaves on X_α*.*

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Theorem 14

Let $\{(U_i, f_i)\}_{i=1}^n$ *be an s-representing distribution. Then there exists a multi-blowup* $\alpha : X_\alpha \to X$ *such that pull-back* $\{(U_i^\alpha, f_i^\alpha)\}_{i=1}^n$ *is solvable i.e. there exists a rational function* f *on* X_α *such that* $f - f_i^\alpha \in \mathcal{O}_{X_\alpha}(U_i^\alpha)$*.*

Proof of Theorem 14

The above lemma yields the following short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{O}_X) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{K}_X) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}_X) \to 0$$

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which induces a long exact sequence in blown-up Čech cohomology

$$0 \to \widetilde{H}^0(\mathcal{U}, \mathcal{O}_X) \to \widetilde{H}^0(\mathcal{U}, \mathcal{K}_X) \to \widetilde{H}^0(\mathcal{U}, \mathcal{H}_X) \to \widetilde{H}^1(\mathcal{U}, \mathcal{O}_X)$$

$$\to \widetilde{H}^1(\mathcal{U}, \mathcal{K}_X) \to \widetilde{H}^1(\mathcal{U}, \mathcal{H}_X) \to \dots$$

By Proposition 6, $\widetilde{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$.

Proof of Theorem 14

Since

$$\widetilde{H}^1(\mathcal{U}, \mathcal{O}_X) = \varinjlim_{\alpha} H^1(\mathcal{U}^\alpha, \mathcal{O}_{X_\alpha}),$$

for any class $\omega \in H^1(\mathcal{U}, \mathcal{O}_X)$ there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^*\omega = 0$ in $H^1(\mathcal{U}^\alpha \mathcal{O}_{X_\alpha})$. To finish the proof it is enough to take $\omega = \zeta(s)$.