Nierówności wielomianowe na zbiorach algebraicznych w $\mathbb{C}^N$

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XXXIX KONFERENCJA I WARSZTATY "GEOMETRIA ANALITYCZNA I ALGEBRAICZNA"

The talk is based on
Białas-Cież L., Calvi J.-P., A. Kowalska, *Polynomial inequalities on certain algebraic hypersurfaces*, J MATH ANAL APPL vol. 459 (2018), 822-838.

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We study extensions of polynomial inequalities (Markov inequality, Schur inequality) and some approximation problems (Pleśniak Theorem, Admissible meshes) to compact subsets of a hypersurface of the form

$$V(f) = \{f(z,y) = 0, (z,y) = (z_1, \dots, z_N, y) \in \mathbb{C}^{N+1}\},$$

where $f(z,y) = y^k - s(z)$, $k \geq 1$ and s is a non constant polynomial in $\mathcal{P}(\mathbb{C}^N)$ and we use y instead of z_{N+1} to emphasize the particular role played by this variable.

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where $f(z,y) = y^k - s(z)$, $k \geq 1$ and s is a non constant polynomial in $\mathcal{P}(\mathbb{C}^N)$ and we use y instead of z_{N+1} to emphasize the particular role played by this variable.

By $\mathcal{P}(\mathbb{C}^N)$ (resp. $\mathcal{P}_d(\mathbb{C}^N)$) we denote the space of all polynomials of N complex variables with coefficients in $\mathbb{C}$ (resp. of total degree at most d). Sometimes, however, it is more convenient to write $\mathcal{P}(z_1, \ldots, z_N)$ or $\mathcal{P}(z)$, $z = (z_1, \ldots, z_N)$ for $\mathcal{P}(\mathbb{C}^N)$ and likewise for the subspaces of polynomials of given degree.

We use standard multinomial notation. In particular, for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, we have $|\alpha| = \alpha_1 + \dots + \alpha_N$, $z^\alpha = z_1^{\alpha_1} \dots z_N^{\alpha_N}$ and $D^\alpha = \partial^{|\alpha|}/(\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N})$.

Definition (Markov set and Markov inequality)

A compact set $E \subset \mathbb{C}^N$ is said to be *a Markov set* if there exist constants $M, m > 0$ such that

$$\|D^\alpha p\|_E \leq M^{|\alpha|}(\deg p)^{m|\alpha|}\|p\|_E, \quad p \in \mathcal{P}(\mathbb{C}^N), \quad \alpha \in \mathbb{N}^N. \quad (1)$$

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By iteration, inequality (1) is satisfied for all α once it is satisfied for all $|\alpha|$ of length one.

Considerable work has been done in the last decades about the problem of finding (geometrical) conditions ensuring that a given compact is a Markov set and that of finding (near) optimal constants in Markov inequalities for a given compact set.

The following properties immediately follow from the definition.

1. A compact set $E \subset \mathbb{C}^N$ is a Markov set if and only if so is $A(E)$ where A is any affine automorphism of $\mathbb{C}^N$.

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3. The Cartesian product of two Markov sets E_i in $\mathbb{C}^{N_i}$, $i = 1, 2$ is a Markov set in $\mathbb{C}^{N_1+N_2}$.
4. A Markov set E in $\mathbb{C}^N$ is $\mathcal{P}(\mathbb{C}^N)$-*determining* (for short *determining*) that is, $p \in \mathcal{P}(\mathbb{C}^N)$ and $\|p\|_E = 0$ implies $p = 0$. (Otherwise, (1) cannot hold for a polynomial p of minimal positive degree which vanishes on E.)

Theorem (Pleśniak Theorem)

A determining compact set E in $\mathbb{C}^N$ is a Markov set if and only if there exist positive constants M and m such that

$$\|p\|_{E_n} \leq M\|p\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N), \quad n \in \mathbb{N} \tag{2}$$

where $E_n = \{z \in \mathbb{C}^N \ : \ \mathrm{dist}(z, E) \leq 1/n^m\}$. The constant m coincides with the exponent m in (1).

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The inequality (2) is called a *Pleśniak Inequality*.

Another important inequality related to Markov inequality is concerned with the problem of bounding a factor of a polynomial in terms of the polynomial itself.

Definition (Division set and division inequality)

A determining compact set E in $\mathbb{C}^N$ is said to be a *division set* if for all non zero polynomial q, we have

$$\|p\|_E \leq D(E, q, n)\|pq\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N), \tag{3}$$

where $D(E, q, n)$ grows polynomially in n.

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The most classical division inequality, which holds for $E = [-1,1]$, is due to Schur and states that

$$\|p\|_{[-1,1]} \leq (1 + \deg p)\|pq\|_{[-1,1]}, \quad q(x) = x, \quad p \in \mathcal{P}(\mathbb{R}).$$

For this reason division inequalities are sometimes called Schur inequalities.

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Theorem

Let E be a compact set in $\mathbb{C}^N$ satisfying Markov inequality (1) and $q \in \mathcal{P}(\mathbb{C}^N)$ a non zero polynomial of degree d. There exists a positive constant C depending only on q and E such that

$$\|p\|_E \leq C(d+n)^{dm}\|pq\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N). \tag{4}$$

We will need a slight extension of this theorem to the case of polynomial vectors.

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Let $\mathbf{P} = (p_1, \dots, p_l)^T$ is a column vector whose entries p_i are polynomials in $\mathcal{P}(\mathbb{C}^N)$ and $\mathbf{A} = (q_{ij})$ is a $l \times l$ matrix whose entries q_{ij} are elements of $\mathcal{P}(\mathbb{C}^N)$. We denote
$\|\mathbf{P}\|_E = \max\{\|p_i\|_E : i = 1, \dots, l\}$ and $\|\mathbf{A}\|_E = \sum_{j=1}^n \|\mathrm{Col}_j(\mathbf{A})\|_E$
where $\mathrm{Col}_j(\mathbf{A})$ denotes the j-th column of $\mathbf{A}$.

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Corollary

Let $E \subset \mathbb{C}^N$ be a compact set in $\mathbb{C}^N$ satisfying Markov inequality (1) and $\mathbf{A}$ be a fixed polynomial matrix as above whose determinant is a non zero polynomial of degree r. Then there exists a positive constant c depending only on $\mathbf{A}$ and E such that

$$\|\mathbf{P}\|_E \leq c(r+n)^{rm}\|\mathbf{AP}\|_E, \quad \mathbf{P} = (p_1, \ldots, p_l)^T, \quad p_i \in \mathcal{P}_n(\mathbb{C}^N). \tag{5}$$

We work with

$$V(f) = \{f(z,y) = 0, (z,y) = (z_1, \ldots, z_N, y) \in \mathbb{C}^{N+1}\},$$

where $f(z,y) = y^k - s(z)$, $k \geq 1$ and s is a non constant polynomial in $\mathcal{P}(\mathbb{C}^N)$

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Baran M., A.K., *Sets with the Bernstein and generalized Markov properties*, Ann. Pol. Math. 111.3(2014), 259-270.

In this paper we consider Bernstein property and generalizations of Markov inequality and Pleśniak condition for compact symetric subsets of algebraic sets of the form

$$\mathbb{V} = \left\{(\widetilde{x}, x_N) \in \mathbb{R}^N : \ x_N^2 = Q(\widetilde{x})\right\},$$

where $\widetilde{x} \in \mathbb{R}^{N-1}$, $Q \in \mathbb{R}[x_1, \dots, x_{N-1}]$ such that $Q^{-1}([0, +\infty)) \neq \emptyset$ and $\deg Q \leq d$.

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A basic but fundamental observation is that f is invariant under the group $\mathbb{U}_k$, that is,

$$f(z, wy) = f(z, y) \text{ for any } w \in \mathbb{U}_k.$$

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$$f(z, wy) = f(z, y) \text{ for any } w \in \mathbb{U}_k.$$

In particular $(z, y) \in V \implies (z, wy) \in V$.

Recall that the ring of polynomials on $V = V(f)$ is

$$\mathcal{P}(V) = \left\{ p_{|V}, p \in \mathcal{P}(z, y) \right\}.$$

We have a very simple algebraic structure for $\mathcal{P}(V)$ as shown by the following lemma and this is one of the two key technical points used in the sequel.

Lemma

We have

$$\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \simeq \mathcal{P}(V).$$

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$$\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \simeq \mathcal{P}(V).$$

As usual, $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ denotes the subspace of $\mathcal{P}(z, y)$ formed of all polynomials of the form

$$\sum_{i=0}^{k-1} c_i(z) y^i \text{ with } c_i \in \mathcal{P}(z).$$

A specific isomorphism

$$\Phi : \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \longrightarrow \mathcal{P}(V)$$

is merely the restriction to V, that is $\Phi(p) = p_{|V}$ while Φ^{-1} is the unique linear map on $\mathcal{P}(V)$ obtained by substituting $s(z)$ for y^k, that is

$$\Phi^{-1}\left((z^\alpha y^m)_{|V}\right) = z^\alpha s^q(z) y^r$$

where $m = qk + r$, $r \in \{0, \ldots, k-1\}$.

The linear map Φ above is one-to-one. Thus, on V, any polynomial coincides with a polynomial from $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$.

We need to suitably define the degree of a polynomial on V. The natural definition (which works for any algebraic set) is as follows.

Definition

The degree $\deg_V p$ of a polynomial $p \in \mathcal{P}(V)$ is defined as

$$\deg_V p = \min \left\{ \deg P : P_{|V} = p \right\}.$$

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In particular, for any $P \in \mathcal{P}(z, y)$, we have

$$\deg_V P_{|V} \leq \deg P.$$

Lemma

For any $p \in \mathcal{P}(V)$ *we have*

$$\deg_V p \leq \deg \Phi^{-1}(p) \leq \max\left\{1, \frac{\deg s}{k}\right\} \deg_V p$$

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Lemma

If $p(z,y) = \sum_{i=0}^{k-1} p_i(z) y^i \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ *then for any* $i = 0, \ldots, k-1$ *and* $(z,y) \in \mathbb{C}^{N+1}$,

$$|p_i(z) y^i| \leq \max_{w \in \mathbb{U}_k} |p(z, wy)|.$$

To explain the way we will use this result we first need the following definition.

Definition

A compact set E in V is said to be $\mathbb{U}_k$-invariant if $(z, y) \in E$ implies $(z, wy) \in E$ for any $w \in \mathbb{U}_k$.

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The whole algebraic set V is $\mathbb{U}_k$-invariant.

Lemma

Let E be a $\mathbb{U}_k$-invariant compact subset of V. If

$$p(z, y) = \sum_{i=0}^{k-1} p_i(z) y^i \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$$

then

$$\|p_i(z) y^i\|_E \leq \|p\|_E, \quad i = 0, \ldots, k-1.$$

One can find similar result for circled sets

Lemma

Let $d \in \mathbb{N}$ and $E \subset C^N$ be a compact, circled set, i.e. $(z, w) \in E$ implies $(e^{it}z, e^{it}w) \in E$ for all real t. Then for any polynomial $p_d = h_d + h_{d-1} + \ldots + h_0$ of degree d written as a sum of homogeneous polynomials, we have $\|h_j\|_E \leq \|p_d\|_E$ for $j = 0, \ldots, d$.

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In the general case, when Z is the set of zeros of a polynomial g in $N + 1$ complex variables, one can ask about a condition which implies a similar estimate for compact subsets of Z.
What about other classes of compact subsets of algebraic varieties with some symmetries which gives us the similar estimation?

Given a compact set E in $V = \{f = 0\}$ as in the previous section, we set

$$|p|^V_{\alpha,E} := \inf\left\{\|D^\alpha P\|_E : P_{|V} = p, P \in \mathcal{P}(z,y)\right\}, \quad p \in \mathcal{P}(V).$$

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$$|p|^V_{\alpha,E} \leq M^{|\alpha|} (\deg_V p)^{m|\alpha|} \|p\|_E, \quad p \in \mathcal{P}(V), \quad \alpha \in \mathbb{N}^N. \qquad (6)$$

This inequality is called a *Markov inequality* for E in V or a *V-Markov inequality*.

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This inequality is called a *Markov inequality* for E in V or a *V-Markov inequality*.

This definition raises evident difficulties as it seems complicated to estimate $|p|^V_{\alpha,E}$.

In fact, we often prove a much stronger inequality in which $|p|^{V}_{\alpha,E}$ is replaced by its upper bound $\|D^{\alpha}\Phi^{-1}(p)\|_{E}$.

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The isomorphism $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \simeq \mathcal{P}(V)$ next suggests the following definition.

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To say that **W** subspace of $\mathcal{P}(\mathbb{C}^{N+1})$ is invariant under derivation simply means that $p \in \mathbf{W}$ implies $D^\alpha p \in \mathbf{W}$ for all α and, of course, it suffices to check the property for $|\alpha| = 1$. The space $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ is obviously invariant by derivation.

Definition (Markov set and Markov inequality on $\mathbf{W}$)

Let $\mathbf{W}$ be an infinite dimensional subspace of $\mathcal{P}(\mathbb{C}^{N+1})$ which is invariant under derivation. A compact set $E \subset \mathbb{C}^{N+1}$ is said to be a $\mathbf{W}$-*Markov set* if there exist $M, m > 0$ such that

$$\|D^\alpha p\|_E \leq M^{|\alpha|}(\deg p)^{m|\alpha|}\|p\|_E, \quad p \in \mathbf{W}, \quad \alpha \in \mathbb{N}^N. \tag{7}$$

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The constant m in (6) and in (7) is called the *exponent* of the respective inequalities.

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$$\|D^\alpha p\|_E \le M^{|\alpha|}(\deg p)^{m|\alpha|}\|p\|_E, \quad p \in \mathbf{W}, \quad \alpha \in \mathbb{N}^N. \qquad (7)$$

This inequality is called a **W**-*Markov inequality* for E.

The constant m in (6) and in (7) is called the *exponent* of the respective inequalities.

Lemma

Let E be a compact subset of V and $\mathbf{W} = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$. If E is a **W***-Markov set then E is also a V-Markov set. The exponent m in the* **W***-Markov inequality may be used in the V-Markov inequality as well.*

From now on, we denote by π the projection from $V \subset \mathbb{C}^{N+1}$ onto the space $\mathbb{C}^N$, i.e. $\pi(z, y) = z$ for $(z, y) \in V$. In particular, if E is a compact subset of V then

$$\pi(E) = \{z \in \mathbb{C}^N : (z, y) \in E \text{ for some } y \in \mathbb{C}\}.$$

Theorem

Let E be a $\mathbb{U}_K$-invariant compact set in V and $\mathbf{W} = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$. Then E is a $\mathbf{W}$-Markov set with $\mathbf{W} = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ if and only if $\pi(E)$ is a Markov set in $\mathbb{C}^N$. In particular, E is a V-Markov set with exponent $m\left(1 + \frac{(k-1)d}{k}\right)$ as soon as $\pi(E)$ is a Markov set with exponent m in $\mathbb{C}^N$.

Example

Let $V = \{y^3 = z^2 - 1\} \subset \mathbb{C}^2$ and $\mathbb{D}$ be the (closed) unit disc in $\mathbb{C}$. The compact set $E = \{(z, y) \in V : y \in \mathbb{D}\}$ is a V-Markov set.

Example

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We have $\pi(E) = \{z \in \mathbb{C} : z^2 - 1 \in \mathbb{D}\}$ which is the lemniscate of Bernoulli (with its interior) and E is $\mathbb{U}_3$-invariant. By a result of Szegö and Bernstein's pointwise estimate we can show that $\pi(E)$ is a Markov set with exponent $m = 1$. Therefore, the set E is a **W**-Markov set and the exponent can be taken as $1 + 2 \cdot (2/3) = 7/3$.

A compact set E in V is $\mathcal{P}_V$-*determining* if for all $p \in \mathcal{P}(z, y)$, $p = 0$ on E implies $p = 0$ on V.

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Definition (Division set and division inequality on V)

A $\mathcal{P}_V$-determining compact subset E in V is said to be a V-division set if, for any non constant polynomial q on V, there exists a sequence $D_V(E, q, n)$ in $\mathbb{R}^+$ which grows polynomially in n such that

$$\|p\|_E \leq D_V(E, q, n)\|pq\|_E, \quad \deg_V p \leq n.$$

Any specific polynomial bound for $D_V(E, q, n)$ is called a *V-division inequality*.

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Theorem

Assume that the polynomial f defining V is irreducible and let E be a $\mathbb{U}_k$-invariant compact set in V. If $\pi(E)$ is a Markov set then E is a V-division set.

This is a particular case of the following result which does not require f to be irreducible.

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Theorem

Let E be a $\mathbb{U}_k$-invariant, $\mathcal{P}_V$-determining compact set in V such that $\pi(E)$ is a Markov set in $\mathbb{C}^N$ and q be a non constant polynomial in $\mathcal{P}(z,y)$. There exists a sequence $D_V(E,q,n)$ that grows polynomially in n such that

$$\|p\|_E \leq D_V(E,q,n)\|pq\|_E \text{ for } \deg_V p \leq n$$

if and only if q and f are relatively prime.

Finally, we can advertise that our results allow us to construct an admissible mesh on $\mathbb{U}_k$-invariant compact subset of V.

Definition

Let E be a compact set in V. A sequence of sets $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in E is called an admissible mesh if:

1. the cardinality of $\mathcal{A}_n$ grows polynomially in n as $n \to \infty$,
2. there exist a positive constant C (independent of n) such that for all $P \in \mathcal{P}(V)$,

$$\|P\|_E \leq C\|P\|_{\mathcal{A}_n} \text{ if } \deg_V P \leq n. \tag{8}$$

We denote by $\pi(K)$ the projection of the set $K \subset \mathbb{C}^{N+1}$ on the space $\mathbb{C}^N$, i.e.

$$\pi(K) := \{z \in \mathbb{C}^N : (z, y) \in K \text{ for some } y \in \mathbb{C}\}. \tag{9}$$

Proposition

If E is a $\mathbb{U}_k$-invariant compact set in V and $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is an admissible mesh for $\pi(E)$ then $\pi^{-1}((\mathcal{A}_n)_{n\in\mathbb{N}})$ is an admissible mesh for E.

Our results concern algebraic hypersurfaces of the form

$$V = \{z_{N+1}^k = s(z_1, \ldots, z_N)\} \subset \mathbb{C}^{N+1}, \tag{10}$$

where s is a non constant polynomial of N variables. We supposed that our results can be generalized by using similar approach to other classes of algebraic varieties.

Thank you for your attention!