Arnold's Problem on monotonicity of Newton numbers

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- $f_0(0) = 0, \quad \nabla f_0(0) := (\frac{\partial f_0}{\partial z_1}(0), \ldots, \frac{\partial f_0}{\partial z_n}(0)) = 0,$
- $\nabla f_0(\mathbf{z}) \neq 0$ for $\mathbf{z} \neq 0$ near 0.

Definitions

The main invariant (topological) of a singularity is the **Milnor number** defined in many ways:
$\mu(f) := \dim_{\mathbb{C}} \mathcal{O}^n / (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$ ($\mathcal{O}^n$ - the ring of all convergent series in n - variables)
=the multiplicity of the mapping $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ at 0,
$(= \max(\#(\nabla f)^{-1}(y),\ y \text{ small}))$
$= \#$(critical points of morsification of f)
$=$ (the topological degree of $\frac{\nabla f}{|\nabla f|} : S_\varepsilon^{2n-1} \to S_1^{2n-1}$)
$= i_0((\nabla f)\mathcal{O}^n)$ – multiplicity of the ideal (∇f) in $\mathcal{O}^n$
$= \mathrm{rk}\, H_{n-1}(F_\theta, \mathbb{Z})$ – (F_θ the fibre of the Milnor fibration of f)

Definitions

For "almost all" singularities $\mu(f)$ can be computed of a combinatoric object associated to f - the **Newton polyhedron**. Let

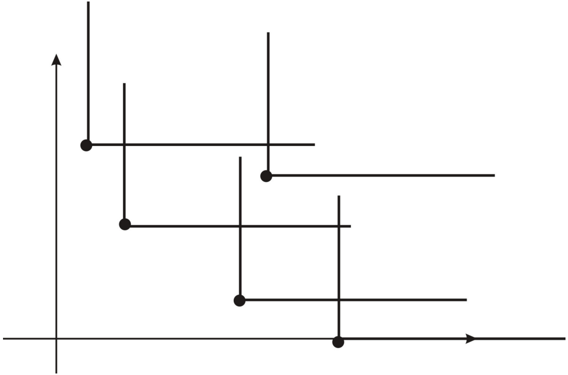
$$f(z) = \sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}} z^{\mathbf{i}}, \quad \mathbf{i} = (i_1, \ldots, i_n), \; z = (z_1, \ldots, z_n)$$

and

$$\mathrm{supp}(f) := \{\mathbf{i} : a_{\mathbf{i}} \neq 0\} \subset \mathbb{R}^n \quad - \text{ the } \textbf{support} \text{ of } f.$$

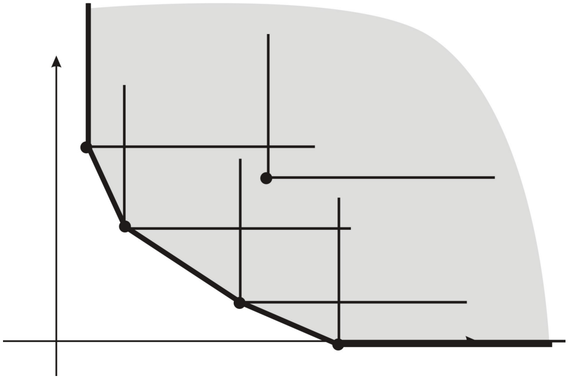
Definitions

$$\bigcup_{\mathbf{i} \in \operatorname{supp}(f)} (\mathbf{i} + \mathbb{R}^n_+)$$



Definitions

$$\Gamma^+(f) := \operatorname{conv}\left(\bigcup_{\mathbf{i}\in \operatorname{supp}(f)} (\mathbf{i}+\mathbb{R}^n_+)\right)$$



Koushnirenko Theorem

Theorem

If f is non-degenerate (generic property) and convenient then

$$\mu(f) = \nu(f),$$

where

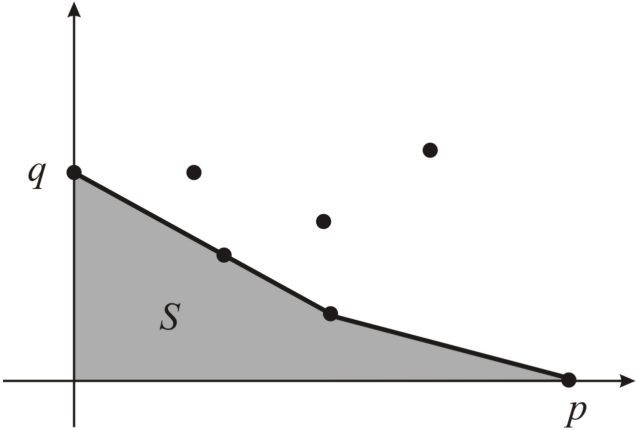
$$\nu(f) := n!V - \sum_{i=1}^{n}(n-1)!V_i + \sum_{i,j=1,i<j}^{n}(n-2)!V_{ij} + \cdots ,$$

where V is n-dimensional volume of the polyhedron under $\Gamma^+(f)$, V_i is $(n-1)$-dimensional volume of the polyhedron under $\Gamma^+(f)$ on the hyperplane $H_i := \{x_i = 0\}$, V_{ij} is $(n-2)$-dimensional volume of the polyhedron under $\Gamma^+(f)$ on the hyperplane $H_{ij} := \{x_i = x_j = 0\}$and so on.

Examples

For instance,

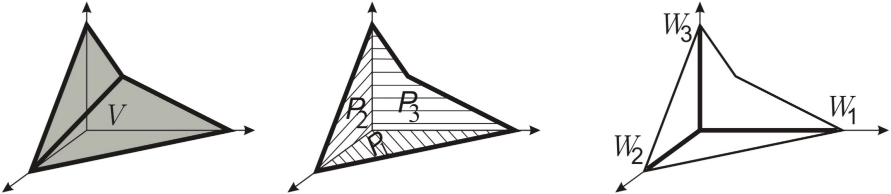
$n = 2$



$$\nu(f) = 2!S - 1!(p+q) + 1$$

Examples

$n = 3$



$$\nu(f) = 3!V - 2!(P_1 + P_2 + P_3) + 1!(W_1 + W_2 + W_3) - 1$$

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- **Remark.** He denotes $\mu(\Delta)$ instead of $\nu(\Delta)$. In the sequel we denote the Newton number by $\nu(\Delta)$.

Known Results

- S.K.Lando (Comments to Arnold's Problems in Arnold's Problems, Springer 2005) wrote that the monotonicity of $\nu(\Delta)$ follows from the semicontinuity of the spectrum of a singularity, proved independently by Varchenko (1983) and Steenbrink (1985).

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- A.Lenarcik, J. Gwoździewicz (2007) gave an elementary proof of monotonicity of $\nu(\Delta)$ for $n = 2$.

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Introduction to results

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Introduction to results

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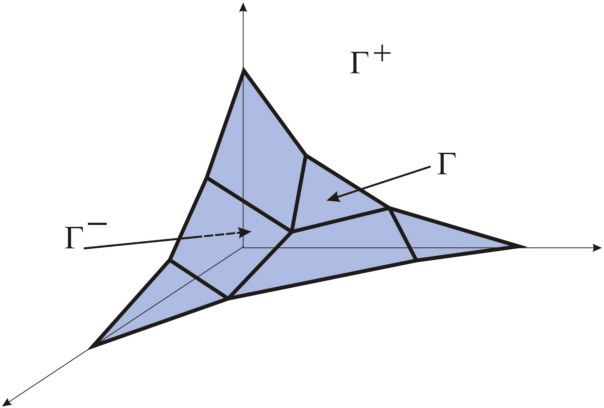
- We give a simple geometrical condition which is necessary and sufficient for the inequality $\nu(\Delta) > \nu(\tilde{\Delta})$ for two Newton polyhedra $\Delta \subset \tilde{\Delta}$.
- The proof is elementary.

Preparation

Let Γ^+ be a convenient Newton polyhedron in $\mathbb{R}^n_+$

$$\Gamma^+ = \operatorname{conv}\left(\bigcup_{\mathbf{i=1}}^{k} (P_i + \mathbb{R}^n_+)\right), \quad P_i \in \mathbb{R}^n_+,$$

$$\Gamma^+ \cap Ox_i \neq \varnothing, \quad i = 1, \ldots, n$$



Γ^+ - Newton polyhedron, Γ - boundary of the Newton polyhedron, Γ^- - polyhedron under the Newton polyhedron, $\Gamma^+, \Gamma, \Gamma^-$ - closed sets.

$$\nu(\Gamma) = n!\,\mathrm{vol}(\Gamma^-) - \sum_{i=1}^{n}(n-1)!\,\mathrm{vol}(\Gamma_i^-) + \sum_{i,j=1, i<j}^{n}(n-2)!\,\mathrm{vol}(\Gamma_{ij}^-) + \cdots ,$$

Preparation

- Let $\tilde{\Gamma}^+$ be another Newton poyhedron such that $\Gamma^+ \subset \tilde{\Gamma}^+$. Then

$$\tilde{\Gamma}^+ = \operatorname{conv}\left(\Gamma^+ \cup \{P_1, \ldots, P_k\}\right),$$

where P_i lie under Γ^+ i.e. $P_i \in \mathbb{R}^n_+ \setminus \Gamma^+$ (the same $P_i \in \Gamma^- \setminus \Gamma$). We will denote

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- Since $\Gamma^+ + \{P_1, P_2\} = (\Gamma^+ + \{P_1\}) + \{P_2\}$, the equality $\nu(\Gamma^+ + \{P_1, P_2\}) = \nu((\Gamma^+ + \{P_1\}) + \{P_2\})$ holds.

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- Then we may restrict consideration to the case when one point is added.

$$\tilde{\Gamma}^+ = \Gamma^+ + \{P\},$$

where P lies under Γ^+ i.e. $P_i \in \mathbb{R}^n_+ \setminus \Gamma^+$.

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Definition

A **pyramid** $\mathbb{P}$ with the base W and the vertex Q is

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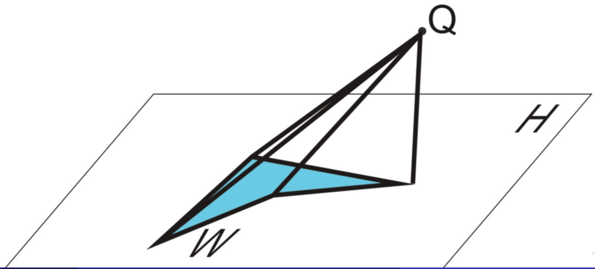
- 

Main result

Theorem

Let Γ^+ *be a convenient Newton polyhedron in* $\mathbb{R}^n_+$ *and let* P *lies under* Γ^+ *i.e.* $P \in \mathbb{R}^n_+ \setminus \Gamma^+$. *Then:*

1. $\nu(\Gamma + P) \leq \nu(\Gamma)$,

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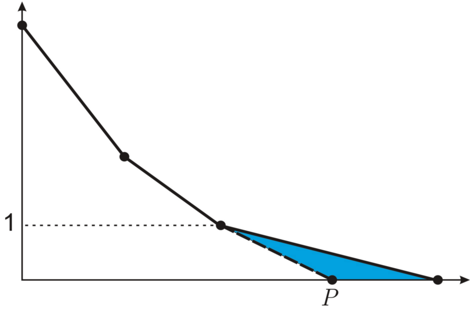
Let Γ^+ be a convenient Newton polyhedron in $\mathbb{R}^n_+$ and let P lies under Γ^+ i.e. $P \in \mathbb{R}^n_+ \setminus \Gamma^+$. Then:

1. $\nu(\Gamma + P) \leq \nu(\Gamma)$,
2. $\nu(\Gamma + P) = \nu(\Gamma)$ *if and only if there exists a coordinate hyperplane* $H = \{x_i = 0\}$ *such that* $P \in H$ *and the difference of these two Newton polyhedra* $\overline{(\Gamma^+ + P) \setminus \Gamma^+}$ *is a pyramid with the base* $\left(\overline{(\Gamma^+ + P) \setminus \Gamma^+}\right) \cap H$ *and the height equal to* 1.

Typical examples

$n = 2$



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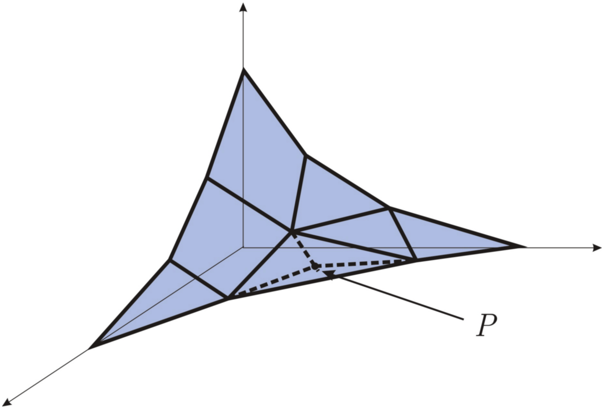

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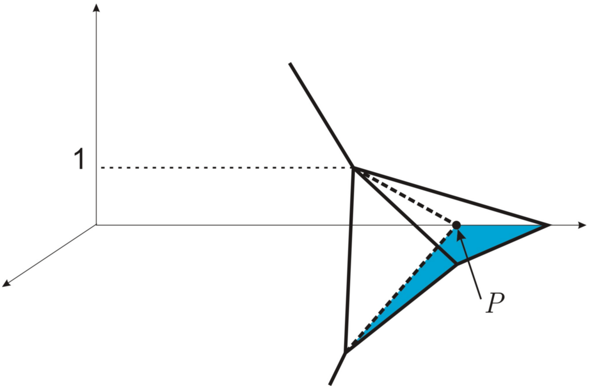

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- $2'$. $\nu(\Gamma + P) < \nu(\Gamma)$ if and only if one of two conditions is fulfilled:
- (a) P lies in the interior of Γ^- i.e. $P \in \operatorname{Int} \Gamma^-$
- (b) for each hyperplane $H = \{x_i = 0\}$ such that $P \in H$ the difference of two Newton polyhedra $\overline{(\Gamma^+ + P) \setminus \Gamma^+}$ is either a pyramid with the base $\left(\overline{(\Gamma^+ + P) \setminus \Gamma^+}\right) \cap H$ and the height ≥ 2 or a polyhedron with the base $\left(\overline{(\Gamma^+ + P) \setminus \Gamma^+}\right) \cap H$ which has at least 2 vertices above H.

Example

Example

Let Γ be the Newton poyhedron of the singularity $f = x^6 + 2y^6 + z(x^2 + y^2) + z^4$, $\nu(\Gamma) = 15$. If we add the point $P = (3, 2, 0)$ we obtain the Newton poyhedron $\Gamma + P$ of the singularity $g = x^6 + 2y^6 + z(x^2 + y^2) + z^4 + x^2y^3$ for which $\nu(\Gamma + P) = 13$. In this case the difference is a polyhedron with the base in the hyperplane $\{z = 0\}$ which has the height equal to 1 but has two vertices above $\{z = 0\}$ (it is not a pyramid).

Key lemma

Lemma

Let Γ^+ *be a convenient Newton polyhedron and let* Γ_n^+ *be the restriction of* Γ^+ *to the hyperplane* $H = \{x_n = 0\}$. *If* P *lies under* Γ^+ *and* $P \in H$ *and the difference of two Newton polyhedra* $\overline{(\Gamma^+ + P) \setminus \Gamma^+}$ *is a pyramid with the base* $\left(\overline{(\Gamma^+ + P) \setminus \Gamma^+}\right) \cap H$ *and the height* h *then*

$$\nu(\Gamma) - \nu(\Gamma + P) = (\nu(\Gamma_n) - \nu(\Gamma_n + P))\,(h - 1).$$

Thank you