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LIMITING CASES OF A UNIVALENCE CRITERION OF HOLOMORPHIC FUNCTIONS

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1. In this article we investigate limiting cases of the following univalence criterion

Theorem 1 ([3]). *Let $a \geq \frac{1}{2}$, $s = \alpha + \beta i$, $\alpha > 0$, $\beta \in \mathbb{R} = (-\infty, \infty)$ be fixed numbers and let $f(z) = z + \dots$ and $g(z)$ be regular in $D = \{z : |z| < 1\}$. If the following inequalities*

$$\left| \frac{zf'(z)}{f(z)g(z)} - \frac{as}{\alpha} \right| \leq \frac{a|s|}{\alpha} \tag{1}$$

and

$$\left| |z|^{\frac{2a}{\alpha}} \frac{zf'(z)}{f(z)g(z)} + \left(1 - |z|^{\frac{2a}{\alpha}}\right) \left[\frac{zf'(z)}{f(z)} + s\frac{zg'(z)}{g(z)} \right] - \frac{as}{\alpha} \right| \leq \frac{a|s|}{\alpha} \tag{2}$$

hold for $z \in D$, then f is univalent in D.

These limiting cases seem to be interesting as they lead to some well-known and important classes of univalent functions as well as to new univalence criteria (see Theorems 5 and 6).

In what follows we need the following important theorem to prove some of our results.

Theorem 2 ([5]). *Let f be Bazilevič function of type (α, β). Then, for each $r \in (0, 1)$,*

$$\int_{\Theta_1}^{\Theta_2} \left\{ 1 + \mathrm{Re} \left[\frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)} \right] - \beta \,\mathrm{Im}\, \frac{zf'(z)}{f(z)} \right\} d\Theta > -\pi, \qquad z = re^{i\Theta},$$

whenever $\Theta_1 < \Theta_2$. *Conversely, if* f *is regular in* D, *with* $f(0) = 0$, $f(z) \neq 0$ $(0 < |z| < 1)$ *and* $f'(z) \neq 0$ *for* $z \in D$, *and if* f *satisfies the last inequality for* $0 < r < 1$, *where* $\alpha \geq 0$ *and* $\beta \in \mathbb{R}$, *then* f *is univalent in* D, *and is Bazilevič function of type* (α, β) *in the case* $\alpha > 0$.

Let us remind that f is Bazilevič function of type (α, β) if for some $\alpha > 0$ it can be represented by the formula

$$f(z) = \left[\int_0^z \frac{p(z)}{p(0)} h^\alpha(z) z^{i\beta - 1} dz\right]^{\frac{1}{\alpha + i\beta}} \qquad \text{for} \quad z \in D,$$

where h, $h(0) = 0$, is a starlike and univalent function in D, p is regular, and has positive real part there ([1]).

In what follows we denote by $B(\alpha, \beta)$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, the class of functions f satisfying the assumption of Theorem 2 and having the usual normalization $f(0) = 0$, $f'(0) = 1$.

2. We come now to the mentioned limiting cases.

a) $a \to \infty$. In this case we obtain the class $B(\alpha, \beta)$ of Bazilevič functions $(\alpha > 0)$. It was considered earlier in [2].

b) $\alpha \to 0$, $\beta = 0$. Then (1) and (2) lead to he following inequalities

$$\left|\frac{zf'(z)}{f(z)g(z)} - a\right| \leq a \qquad \text{and} \qquad \left|\frac{zf'(z)}{f(z)} - a\right| \leq a,$$

which determine some subclass of the class S^* of functions of the form $f(z) = z + \ldots$, that are univalent and starlike with respect to the origin in D.

c) $\alpha \to 0$, $\beta \to \beta_0 \neq 0$. Relations (1) and (2) lead to the following inequalities

$$\text{(3)} \qquad \operatorname{Re}\left[\frac{1}{i\beta_0} \frac{zf'(z)}{f(z)g(z)}\right] \geq 0$$

and

$$\text{(4)} \qquad \operatorname{Re}\left[\frac{1}{i\beta_0} \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z)}\right] \geq 0$$

for $z \in D$, respectively.

Now, we shall prove the following

Theorem 3. *Let* $f(z) = z + \ldots$ *and* $g(z)$ *be regular in* D *with* $f'(z) \neq 0$ *for* $z \in D$, *and let (3) and (4) hold in* D. *Then* $f \in B\left(0, -\frac{1}{\beta_0}\right)$.

Proof. In what follows let us put $\beta_0 = \beta \in \mathbb{R} - \{0\}$. From (3) we obtain

$$\text{(5)} \quad \frac{1}{i\beta} \frac{zf'(z)}{f(z)g(z)} = p(z), \qquad p(0) = \frac{1}{i\beta g(0)}, \quad p(0) \neq 0, \quad \operatorname{Re} p(z) \geq 0 \quad \text{for} \quad z \in D.$$

Inequality (4), according to maximum principle, gives

$$\text{(6)} \qquad \frac{1}{i\beta} \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z)} = \frac{1}{i\beta}, \qquad z \in D.$$

We deduce from (5) and (6) that

$$\text{(7)}\qquad \frac{1}{i\beta}\frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \frac{zp'(z)}{p(z)} = \frac{1}{i\beta}, \qquad z \in D.$$

It follows from (7) and by $\operatorname{Re} p(z) \geq 0$ in D and $p(0) \neq 0$ that

$$\text{(8)}\qquad \int_{\Theta_1}^{\Theta_2} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{1}{i\beta}\frac{zf'(z)}{f(z)}\right\} d\Theta > -\pi,$$
$$z = re^{i\Theta}, \quad r \in (0,1),$$

whenever $\Theta_1 < \Theta_2$. Our assertion follows by Theorem 2 for $\alpha = 0$.

d) $\alpha \to 0$, $\beta \to \pm\infty$. Inequalities (1) and (2) lead to the relation (7) and for $\beta \to \pm\infty$ we obtain the limiting case of (8), which by Theorem 2 for $\alpha = 0$ proves that $f \in B(0,0)$.

Remark ([5]). $f \in B(0,\beta)$ if and only if

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{i\beta} = h(z),$$

where $\operatorname{Re}(e^{i\lambda}h(z)) > 0$ for $z \in D$ and some $\lambda \in \mathbb{R}$.

Hence $B(0,0) = \check{S}$ denotes the class of univalent and spiral-like functions of Špaček ([6]).

e) $\alpha \to 0$, $\beta \to 0$.

1^0 $\frac{\beta}{\alpha} \to \pm\infty$. Then (1) and (2) lead to $\operatorname{Re}\left\{\mp i\frac{zf'(z)}{f(z)g(z)}\right\} \geq 0$ and $\operatorname{Re}\left\{\mp i\frac{zf'(z)}{f(z)}\right\} \geq 0$ for $z \in D$, respectively, which, in turn, according to the minimum principle of harmonic functions, determine the function $f(z) = z$ in D.

2^0 $\alpha \to 0$, $\frac{\beta}{\alpha} \to \beta_1 \neq 0$. From (1) we obtain

$$\left|e^{i\gamma}\frac{zf'(z)}{af(z)} - \sqrt{1+\beta_1^2}\right| \leq \sqrt{1+\beta_1^2}, \qquad \gamma = \arg(1+i\beta_1).$$

This inequality determines immediately a subclass of $\check{S}$.

3^0 $\alpha \to 0$, $\frac{\beta}{\alpha} \to 0$. Then $\frac{|s|}{\alpha} \to 1$ and from (1) and (2) we obtain

$$\left|\frac{zf'(z)}{f(z)g(z)} - a\right| \leq a \qquad \text{and} \qquad \left|\frac{zf'(z)}{f(z)} - a\right| \leq a.$$

Thus in this case we obtain a subclass of S^*.

3. Now let $\alpha > 0$ be a fixed number and $\beta \to \pm\infty$. Then (1) and (2) lead to

$$g(z) = \pm ip(z)\frac{zf'(z)}{f(z)} \quad \text{and} \quad \left|\left(1 - |z|^{\frac{2a}{\alpha}}\right)\frac{zg'(z)}{g(z)} - \frac{a}{\alpha}\right| \leq \frac{a}{\alpha} \quad \text{for } z \in D,$$

respectively, where $\operatorname{Re} p(z) \geq 0$, $z \in D$. Hence we obtain by simple calculation

$$(9) \qquad \left| \left(1 - |z|^{\frac{2a}{\alpha}}\right) \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)} - \frac{a}{\alpha}\right) \right| \leq \frac{a}{\alpha} \qquad \text{for} \quad z \in D.$$

This inequality, in turn, implies the following one

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)}\right) \geq 0 \qquad \text{for} \quad z \in D.$$

Hence we obtain for $\Theta_1 < \Theta_2$

$$\int_{\Theta_1}^{\Theta_2} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)}\right) d\Theta > -\pi, \quad z = re^{i\Theta}, \quad r \in (0,1).$$

Thus by Theorem 2 and by Remark $f \in \check{S}$.

4. In this section we will consider the case $\alpha \to \infty$. There are two following possibilities.

a) $\alpha \to \infty$, $\frac{\beta}{\alpha} \to \pm\infty$. Then (1) leads to the equality $g(z) = \pm ip(z)\frac{zf'(z)}{f(z)}$ where $\operatorname{Re} p(z) \geq 0$ in D. Multiplying both sides of (2) by $\frac{\alpha}{|\beta|}$ and considering the equality

$$\lim_{\alpha \to \infty} \left[\alpha(1 - |z|^{\frac{2a}{\alpha}})\right] = -2a \ln|z|, \qquad |z| < 1,$$

we obtain from (2) in the limit the following inequality

$$\left|1 + 2\ln|z| \frac{zg'(z)}{g(z)}\right| \leq 1.$$

From the above considerations we eventually obtain

$$\left|1 + 2\ln|z| \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)}\right)\right| \leq 1 \qquad \text{for} \quad z \in D.$$

This inequality is true if and only if f satisfies the following differential equation

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)} = 0 \qquad \text{for} \quad z \in D.$$

It is easy to verify that each solution f of this equation is an element of S^*.

b) $\alpha \to \infty$, $\frac{\beta}{\alpha} \to \beta_0$. Then $\frac{S}{\alpha} = 1 + i\frac{\beta}{\alpha} \to 1 + i\beta_0$. Multiplying both sides of (1) and (2) by $\frac{\alpha}{|\beta|}$ and considering the equality

$$\lim_{\alpha \to \infty} \left[\alpha\left(1 - |z|^{\frac{2a}{\alpha}}\right)\right] = -2a \ln|z|, \qquad |z| < 1,$$

we obtain from (1) and (2) in the limit the following inequalities

$$\left|e^{-i\varphi_0} \frac{zf'(z)}{f(z)g(z)} - a\sqrt{1 + \beta_0^2}\right| \leq a\sqrt{1 + \beta_0^2}$$

and

$$\left|e^{-i\varphi_0}\frac{zf'(z)}{f(z)} - 2a\sqrt{1+\beta_0^2}\frac{zg'(z)}{g(z)}\ln|z| - a\sqrt{1+\beta_0^2}\right| \le a\sqrt{1+\beta_0^2}$$

respectively, where $\varphi_0 = \arg(1+\beta_0)$ and $z \in D$.

If we put $g_1(z) = ae^{i\varphi_0}\sqrt{1+\beta_0^2}g(z)$ and denote g_1 again by g, then we obtain from the last inequalities the following ones

$$\left|\frac{zf'(z)}{f(z)g(z)} - 1\right| \le 1 \tag{10}$$

and

$$\left|\frac{zf'(z)}{f(z)g(z)} - 2\ln|z|\frac{zg'(z)}{g(z)} - 1\right| \le 1, \tag{11}$$

respectively, for $z \in D$.

These inequalities represent a univalence criterion of f in D but it is not evident immediately. It will be proved in the next section (Theorem 4).

5. We come now to the formulation and proof of

Theorem 4. *Let $f(z) = z + \dots$ and $g(z)$ be regular in D with $f'(z) \neq 0$ there, and let inequalities (10) and (11) hold for $z \in D$. Then f is univalent in D.*

Proof. It follows by (10) that $g(z) \neq 0$ for $z \in D$ and $\left|\frac{1}{g(0)} - 1\right| \le 1$ or $\operatorname{Re} g(0) \ge \frac{1}{2}$. Let us first assume that $\operatorname{Re} g(0) = \frac{1}{2}$. Then we deduce by (10) that left-hand side of (10) attains its maximum at the point $z = 0$. Thus according to maximum principle we obtain $\frac{zf'(z)}{f(z)g(z)} = \frac{1}{g(0)}$ for $z \in D$. Combining this with (11) we have

$$\left|\frac{1}{g(0)} - 2\ln|z|\frac{zg'(z)}{g(z)} - 1\right| \le 1.$$

But $\frac{1}{g(0)} - 1 = e^{i\varphi}$ for some $\varphi \in \mathbb{R}$. Therefore the last inequality may be written in the following form

$$\left|e^{i\varphi} - 2\ln|z|\frac{zg'(z)}{g(z)}\right| \le 1.$$

This inequality can be true if and only if $g(z) = g(0)$ for $z \in D$. Therefore $\frac{zf'(z)}{f(z)} = 1$ in D or $f(z) \equiv z$. Thus Theorem 4 is true in the considered case.

Now we assume that $\operatorname{Re} g(0) > \frac{1}{2}$ and define the following family of functions

$$f(z,t) = f(ze^{-t})\exp[2tg(ze^{-t})], \qquad z \in D, \quad t \in \langle 0, \infty).$$

If $f(z,t) = a(t)z + \dots$, then $a(t) = e^{t(2g(0)-1)}$. Hence $|a(t)| = e^{t(2\operatorname{Re} g(0)-1)}$ and $|a(t)| \to \infty$. Moreover, $\frac{f(z,t)}{a(t)} \to z$ as $t \to \infty$ local uniformly in D. Therefore the family $\left\{\frac{f(z,t)}{a(t)}\right\}$ is normal in D.

Now by simple calculation we obtain

$$A(z,t)=\frac{f'_t(z,t)}{zf'_z(z,t)}=-1+2g(ze^{-t})\left[ze^{-t}\frac{f'(ze^{-t})}{f(ze^{-t})}+2tze^{-t}g'(ze^{-t})\right]^{-1}.$$

Hence we obtain

$$\left|\frac{A(z,t)-1}{A(z,t)+1}\right|=\left|\frac{ze^{-t}f'(ze^{-t})}{f(ze^{-t})g(ze^{-t})}+2t\frac{ze^{-t}g'(ze^{-t})}{g(ze^{-t})}-1\right|.$$

Taking $ze^{-t}=\zeta$ in the last equality we obtain for $|z|=1$ and $t\geq 0$ the following one

$$\left|\frac{A(e^{i\Theta},t)-1}{A(e^{i\Theta},t)+1}\right|=\left|\frac{\zeta f'(\zeta)}{f(\zeta)g(\zeta)}-2\ln|\zeta|\frac{\zeta g'(\zeta)}{g(\zeta)}-1\right|.$$

Hence by (11) we obtain

$$\left|\frac{A(z,t)-1}{A(z,t)+1}\right|\leq 1 \qquad \text{for} \quad z\in D, \quad t\in\langle 0,\infty).$$

From the assumption $\operatorname{Re} g(0)>\frac{1}{2}$ we deduce that $\left|\frac{A(0,t)-1}{A(0,t)+1}\right|=\left|1-\frac{1}{g(0)}\right|<1$. Then according to maximum principle we eventually obtain

$$\left|\frac{A(z,t)-1}{A(z,t)+1}\right|< 1 \qquad \text{for} \quad z\in D, \quad t\in\langle 0,\infty).$$

This inequality tells us that the chain $f(z,t)$ satisfies the equation

$$\frac{\partial f}{\partial t}=z\frac{\partial f}{\partial z}p(z,t)$$

with a function $p(z,t)$ which is regular in D for each $t\in\langle 0,\infty)$ and has positive real part there. It turns out from the above considerations that $f(z,t)$ satisfies all the assumptions of the famous lemma of Pommerenke (Lemma 3, [4]). Hence $f(z,t)$ is univalent in D for each $t\in\langle 0,\infty)$ by this lemma and so is $f(z)=f(z,0)$. The proof of Theorem 4 has been completed.

It is easy to see that (10) implies the following equality

$$\frac{zf'(z)}{f(z)g(z)}=1-\omega(z), \qquad \text{where} \quad |\omega(z)|\leq 1, \quad \omega(z)\neq 1 \quad \text{for} \quad z\in D.$$

Combining this with (11) we obtain the following, equivalent to Theorem 4,

Theorem 5. *Let $f(z)=z+\dots$ and $\omega(z)$ be regular in D with $f'(z)\neq 0$, $|\omega(z)|\leq 1$ and $\omega(z)\neq 1$ there. If the following inequality*

$$\left|\omega(z)+2\ln|z|\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\frac{z\omega'(z)}{1-\omega(z)}\right)\right|\leq 1 \tag{12}$$

hold for $z \in D$, then f is $\widehat{in}$ D.

We come to the next section which contains concluding remarks.

6. It is worth emphasizing that all the considered limiting cases of Theorem 1, excluding the last one, determined functions belonging to $B(0,\beta)$. Now we will prove that there exists a function f which satisfies the assumptions of Theorem 5 and such that $f \notin B(0,\beta)$ for any $\beta \in \mathbb{R}$.

To prove this, first we consider a function f of the form

$$f(z) = z \exp\left\{\int_0^z \frac{1}{z}\left[(1-\omega(z))\exp\left(\int_0^z \frac{\varphi(z)}{z}dz\right) - 1\right]dz\right\}, \tag{13}$$

where ω and φ are regular in D and such that $\omega(0) = \varphi(0) = 0$, $|\omega(z)| < 1$, $|\varphi(z)| < \frac{1}{2}$ for $z \in D$. The function f is univalent in D because it satisfies the assumptions of Theorem 5. Essentially, from (13) we have

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{z\omega'(z)}{1-\omega(z)} = \varphi(z), \qquad z \in D.$$

Hence by Schwarz Lemma we obtain

$$\begin{aligned}&\left|\omega(z) + 2\ln|z|\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{z\omega'(z)}{1-\omega(z)}\right)\right| = |\omega(z) + 2\ln|z|\varphi(z)| \\ &\quad \le |z| - |z|\ln|z| \le 1 \qquad \text{for } \ z \in D.\end{aligned}$$

Thus relation (12) is satisfied.

Let us remind that $f \in B(0,\beta)$ satisfies the following inequality $L(f,\beta,r,\Theta_1,\Theta_2) > -\pi$ whenever $\Theta_1 < \Theta_2$ and $r \in (0,1)$, where

$$\begin{aligned}&L(f,\beta,r,\Theta_1,\Theta_2) \\ &\quad = \int_{\Theta_1}^{\Theta_2}\left[\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) - \beta \operatorname{Im}\frac{zf'(z)}{f(z)}\right]d\Theta, \qquad z = re^{i\Theta}.\end{aligned}$$

Let now f be given by (13), where we assume additionally that φ and ω have real Taylor's coefficients. Then for each symmetric interval $\langle -\Theta, \Theta\rangle$, we have

$$L(f,\beta,r,-\Theta,\Theta) = \int_{-\Theta}^{\Theta} \operatorname{Re}\left[\frac{z\omega'(z)}{1-\omega(z)} + \varphi(z)\right]d\Theta, \qquad z = re^{i\Theta}.$$

Now let $\varphi(z) = -\frac{1}{2}z$, $z \in D$, and let ω be schlicht mapping of D onto the simply connected domains bounded by the circle $|z| = 1$ and the circle whose diameter is the segment $\langle a, 1\rangle$, $0 < a < 1$ (a circular lune). Moreover, let $\omega(-1) = -1$, $\omega(0) = 0$ and $\omega(1) = a$. It is easy to see that $L(f,\beta,r,-\Theta,\Theta) = -\pi - \sin\Theta$ for $|z| = 1$, where $\Theta = \Theta(a) \in (0,\pi)$. Thus we can find $r \in (0,1)$, and such that $L(f,\beta,r,-\Theta,\Theta) < -\pi$ for $|z| = r$ by obvious continuity considerations. It follows that $f \notin B(0,\beta)$ for any $\beta \in \mathbb{R}$.

Additionally, we inform that $\omega(z) = 1 + \frac{1}{H_a(z)}$ and $\Theta = \Theta(a) = \pi\frac{1-a}{1+a}$, where $H_a(z) = \frac{-i}{2\pi}\frac{1+a}{1-a}\ln\frac{1-ze^{i\Theta(a)}}{1-ze^{-i\Theta(a)}}$ maps univalently D onto a suitable vertical strip.

In the last proof this concrete representation of ω was not quite necessary.

Finally, let us observe that Theorem 5 implies the following

Theorem 6. *Let* $F(\zeta)=\zeta+b_0+b_1\zeta^{-1}+\dots$, $F'(\zeta)\neq 0$, *be regular in* $D^0\backslash\{\infty\}$ *and let* $w(\zeta)$, $|w(\zeta)|\le 1$, $w(\zeta)\neq 1$, *be regular in* D^0, *where* $D^0=\{\zeta:|\zeta|>1\}$. *If the following inequality*

$$\left|w(\zeta)+2\ln|\zeta|\left(1+\frac{\zeta F''(\zeta)}{F'(\zeta)}-\frac{\zeta F'(\zeta)}{F(\zeta)}+\frac{\zeta w'(\zeta)}{1-w(\zeta)}\right)\right|\le 1$$

holds for $\zeta\in D^0$, *then* F *is univalent in* D^0.

Essentially, let us put $F(\zeta)=\frac{1}{f(z)}$ and $w(\zeta)=w\left(\frac{1}{z}\right)=\omega(z)$, where $z=\frac{1}{\zeta}$. Then f satisfies all the assumptions of Theorem 5, thus f is univalent in D. Hence we obtain our assertion.

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Przypadki graniczne pewnego kryterium jednolistności

Streszczenie. Znane jest ([3]) następujące kryterium jednolistności

Twierdzenie. *Niech* $a\ge\frac{1}{2}$, $s=\alpha+\beta i$, $\alpha>0$, $\beta\in\mathbb{R}=(-\infty,\infty)$ *będą ustalonymi liczbami i niech* $f(z)=z+\dots$ *i* $g(z)$ *będą funkcjami regularnymi w* $D=\{z:|z|<1\}$. *Jeżeli dla* $z\in D$ *prawdziwe są nierówności*

$$\left|\frac{zf'(z)}{f(z)g(z)}-\frac{as}{\alpha}\right|\le\frac{a|s|}{\alpha}$$

i

$$\left||z|^{\frac{2a}{\alpha}}\frac{zf'(z)}{f(z)g(z)}+\left(1-|z|^{\frac{2a}{\alpha}}\right)\left[\frac{zf'(z)}{f(z)}+s\frac{zg'(z)}{g(z)}\right]-\frac{as}{\alpha}\right|\le\frac{a|s|}{\alpha},$$

to f *jest jednolistna w* D.

W artykule omówione są przypadki graniczne wartości parametrów a, α oraz β. Są one interesujące, gdyż otrzymujemy dobrze znane klasy jednolistnych funkcji lub też nowe kryteria jednolistności.

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