Type 9 points on Fermat cubic

XLIV Analytic and Algebraic Geometry Conference

Łukasz Merta

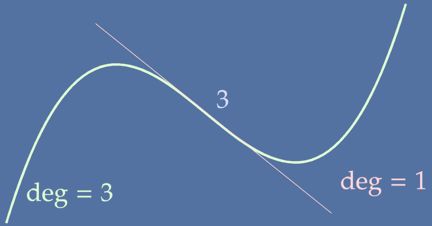
Łódź, January 11, 2024

Flex Points

Points in which the tangent line intersect the curve with multiplicity 3.

If f defines the curve, flex points are common zeroes of f and $H(f)$.

In case $f = x^3 + y^3 + z^3$ we have $H(f) = xyz$ (up to a scalar).



Flex points on Fermat cubic

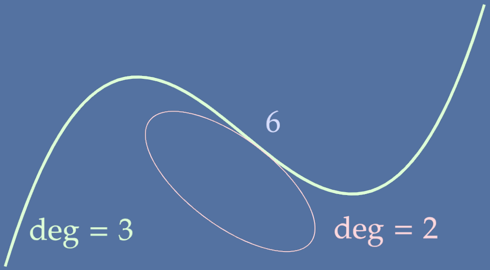
Explicit coordinates of flex points on Fermat cubic:

$$\begin{aligned} P_1 &= [1 : -1 : 0], & P_4 &= [1 : -\varepsilon : 0], & P_7 &= [1 : -\varepsilon^2 : 0], \\ P_2 &= [1 : 0 : -1], & P_5 &= [1 : 0 : -\varepsilon], & P_8 &= [1 : 0 : -\varepsilon^2], \\ P_3 &= [0 : 1 : -1], & P_6 &= [0 : 1 : -\varepsilon], & P_9 &= [0 : 1 : -\varepsilon^2], \end{aligned}$$

where $\varepsilon \in \mathbb{C}$ is a primitive root of unity of order 3.

Sextactic Points

Points for which there exists an irreducible hyperosculating conic with a tangency order of at least 6 at the point.



Sextactic points on Fermat cubic

Sextactic points on Fermat cubic are common zeroes of $f = x^3 + y^3 + z^3$ and

$$H_2(f) = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

Sextactic points on Fermat cubic

Sextactic points on Fermat cubic are common zeroes of $f = x^3 + y^3 + z^3$ and

$$H_2(f) = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

Explicit coordinates of some of sextactic points on Fermat cubic:

$$\begin{aligned} S_1 &= [-\tfrac{1}{2}\varepsilon\mu^2 : -\tfrac{1}{2}\varepsilon\mu^2 : 1], & S_5 &= [\varepsilon : -\varepsilon\mu : 1], \\ S_2 &= [1 : (\varepsilon+1)\mu : 1], & S_6 &= [-\mu : -\varepsilon - 1 : 1], \\ S_3 &= [(\varepsilon+1)\mu : 1 : 1], & S_7 &= [-\tfrac{1}{2}\mu^2 : \tfrac{1}{2}(\varepsilon+1)\mu^2 : 1], \\ S_4 &= [\tfrac{1}{2}(\varepsilon+1)\mu^2 : -\tfrac{1}{2}\mu^2 : 1], & S_8 &= [-\varepsilon - 1 : -\mu : 1], \end{aligned}$$

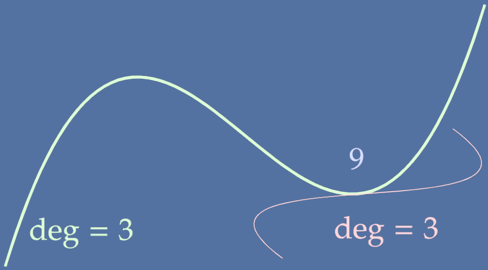
where $\mu = \sqrt[3]{2}$ and $\varepsilon \in \mathbb{C}$ is a primitive root of unity of order 3.

Type 9 Points

Type 9 points are points where an irreducible cubic curve intersects C with multiplicity 9.

Questions:

1. What are the coordinates of type 9 points on Fermat cubic?
2. Is there a curve of degree 24 passing through all of those type 9 points?



Abel's Theorem

Theorem

Let E be a smooth complex elliptic curve with distinguished point 0 embedded in the complex projective plane. Then a divisor $D = \sum d_i P_i$ on E is a (scheme theoretic) intersection of E with another plane curve C of degree c if and only if $\sum d_i = 3c$ and $\sum d_i P_i = 0$ in the group law on E.

Abel's Theorem

Theorem

Let E be a smooth complex elliptic curve with distinguished point 0 embedded in the complex projective plane. Then a divisor $D = \sum d_i P_i$ on E is a (scheme theoretic) intersection of E with another plane curve C of degree c if and only if $\sum d_i = 3c$ and $\sum d_i P_i = 0$ in the group law on E.

Corollaries:

1. Sextactic points on an elliptic curve are 6-torsion points which are not 3-torsion points.
2. Type 9 points on an elliptic curve are 9-torsion points which are not 3-torsion points.

Division polynomials for $y^2 = x^3 + Ax + B$

A sequence $(\psi_n)_{n\in\mathbb{N}}$ of polynomials, defined as:

$$\begin{aligned}
\psi_0 &= 0,\\
\psi_1 &= 1,\\
\psi_2 &= 2y,\\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2,\\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3),\\
\psi_{2n+1} &= \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3 \text{ for } n \geq 2,\\
\psi_{2n} &= \frac{\psi_n}{2y}(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2) \text{ for } n > 2.
\end{aligned}$$

Division polynomials for $y^2 = x^3 + Ax + B$

Let E be an elliptic curve $y^2 = x^3 + Ax + B$.

The roots of ψ_{2n+1} are the x-coordinates of the points of $E[2n+1] \setminus \{O\}$, where $E[2n+1]$ denotes the $(2n+1)$-th torsion group of E.

The roots of $\frac{\psi_{2n}}{y}$ are the x-coordinates of the points of $E[2n] \setminus E[2]$.

Division polynomials for $y^2 = x^3 + Ax + B$

Fermat cubic in Weierstrass form:

$$x^3 + y^3 + z^3 = 0 \longrightarrow y^2 = x^3 - 432$$

$$[x : y : z] \longrightarrow \left(\frac{-12z}{x+y}, \frac{36(x-y)}{x+y} \right)$$

Division polynomials for $y^2 = x^3 + Ax + B$

Fermat cubic in Weierstrass form:

$$x^3 + y^3 + z^3 = 0 \longrightarrow y^2 = x^3 - 432$$

$$[x : y : z] \longrightarrow \left(\frac{-12z}{x+y}, \frac{36(x-y)}{x+y} \right)$$

Polynomial ψ_9 splits in $\mathbb{Q}(\alpha, \beta)$, where $\alpha = \sqrt[3]{3}$ and β is the primitive root of unity of order 9.

Minimal polynomial of $\gamma = \alpha + \beta$:

$$\gamma^{18} - 15\gamma^{15} + 177\gamma^{12} - 578\gamma^{9} + 6747\gamma^{6} + 642\gamma^{3} + 343 = 0.$$

Type 9 points: complete intersection

Theorem.

The set of type 9 points on Fermat cubic is a complete intersection of curves $x^3 + y^3 + z^3 = 0$ and $H_3(y, z) = 0$, where

$$\begin{aligned} H_3(y,z) &= y^{24} + 4y^{21}z^3 - 17y^{18}z^6 - 65y^{15}z^9 - 89y^{12}z^{12} \\ &\quad - 65y^9z^{15} - 17y^6z^{18} + 4y^3z^{21} + z^{24} \\ &= (y^9 + 6y^6z^3 + 3y^3z^6 - z^9)(y^9 - 3y^6z^3 - 6y^3z^6 - z^9)(y^6 + y^3z^3 + z^6). \end{aligned}$$

$H_3(y, z)$ splits into 24 lines passing through $[1 : 0 : 0]$ over $\mathbb{Q}(\alpha, \beta)$.

Coordinates of the first 12 type 9 points

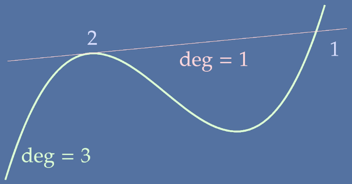
$$P_1 = [1 : \beta : \beta^2], \quad P_2 = [1 : \beta^2 : \beta^4], \quad P_3 = [1 : \beta : \beta^5]$$

Coordinates of the first 12 type 9 points

$$P_1 = [1 : \beta : \beta^2], \quad P_2 = [1 : \beta^2 : \beta^4], \quad P_3 = [1 : \beta : \beta^5]$$

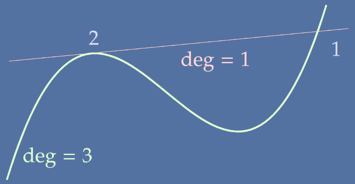
$$\begin{aligned}
P_4 &= [3 : \alpha\beta(\beta^4 + 2\beta^3 - \beta + 1) : -\alpha^2(\beta^2 + \beta + 1)(\beta^3 - \beta + 1)], \\
P_5 &= [3 : -\alpha\beta(2\beta^4 + \beta^3 + \beta + 2) : \alpha^2(\beta + 1)(\beta - 1)^2], \\
P_6 &= [3 : \alpha\beta(\beta^4 - \beta^3 - \beta - 2) : -\alpha^2\beta^2(\beta^2 + \beta + 1)], \\
P_7 &= [3 : \alpha\beta(\beta - 1)(\beta^3 + 2) : -\alpha^2\beta(\beta^2 + \beta + 1)(\beta^2 - \beta + 1)], \\
P_8 &= [3 : \alpha^2(\beta - 1)(\beta + 1)(\beta^3 + \beta + 1) : \alpha\beta(\beta^4 - \beta^3 - \beta - 2)], \\
P_9 &= [3 : \alpha\beta(\beta - 1)^2(\beta^2 + \beta + 1) : -\alpha^2\beta(\beta^2 + \beta + 1)(\beta^2 - \beta + 1)], \\
P_{10} &= [3 : \alpha\beta(\beta - 1)^2(\beta^2 + \beta + 1) : \alpha^2(\beta - 1)(\beta^3 + \beta^2 + 1)], \\
P_{11} &= [3 : \alpha\beta(\beta^4 + 2\beta^3 + 2\beta + 1) : -\alpha^2\beta^2(\beta^2 + \beta + 1)], \\
P_{12} &= [3 : -\alpha^2(\beta^2 + \beta + 1)(\beta^3 - \beta + 1) : -\alpha\beta(2\beta^4 + \beta^3 + \beta + 2)].
\end{aligned}$$

Tangents to Fermat cubic



We now consider lines which are tangent to Fermat cubic at flex points, sextactic points and in type 9 points.

Tangents to Fermat cubic



We now consider lines which are tangent to Fermat cubic at flex points, sextactic points and in type 9 points.

Remark.

Every line tangent to a Fermat cubic at a flex point intersects the cubic in another flex point.

Tangents to Fermat cubic

Theorem.

Every line tangent to a Fermat cubic at a sextactic point intersects the cubic again in a flex point. For each flex point there are 3 lines tangent at sextactic points passing through it.

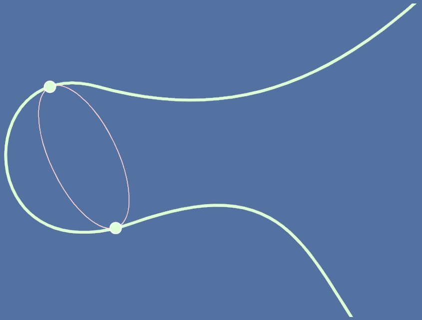
Tangents to Fermat cubic

Theorem.
Every line tangent to a Fermat cubic at a sextactic point intersects the cubic again in a flex point. For each flex point there are 3 lines tangent at sextactic points passing through it.

Theorem.
Every line tangent to a Fermat cubic at a point of type 9 intersects the cubic again in another point of type 9. Moreover, for each point P of type 9 there exists two other points Q, R of type 9 such that:
- line tangent to Fermat cubic at P intersects the cubic in Q,
- line tangent to Fermat cubic at Q intersects the cubic in R,
- line tangent to Fermat cubic at R intersects the cubic in P.

Conics and Type-9 Points

We explore conics that intersect the Fermat cubic at two type-9 points with an intersection multiplicity of 3. These conics have interesting properties.



How to find such conics?

Lemma

If two affine curves f and g, passing through $(0,0)$, can be written as

$$f(x,y) = x + \textit{other monomials}$$
$$g(x,y) = y^k + \textit{monomials of higher degrees},$$

then the intersection index of f and g at $(0,0)$ is at most k.

Algorithm (sketch)

- Let F be the Fermat cubic. Let C be a conic of general form:

$$C : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

- Select a type-9 point P, write its coordinates as $P = [1 : y_P : z_P]$.
- Put $x = 1$, $y = y + y_P$ and $z = z + z_P$ in F and C.
- *Condition 1*: C has to pass through $(0, 0)$.

Algorithm (sketch)

- Let F be the Fermat cubic. Let C be a conic of general form:

$$C : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

- Select a type-9 point P, write its coordinates as $P = [1 : y_P : z_P]$.
- Put $x = 1$, $y = y + y_P$ and $z = z + z_P$ in F and C.
- *Condition 1*: C has to pass through $(0, 0)$.
- "Remove" monomial z from C.
- *Condition 2*: the coefficient at y in C has to be equal to 0.

Algorithm (sketch)

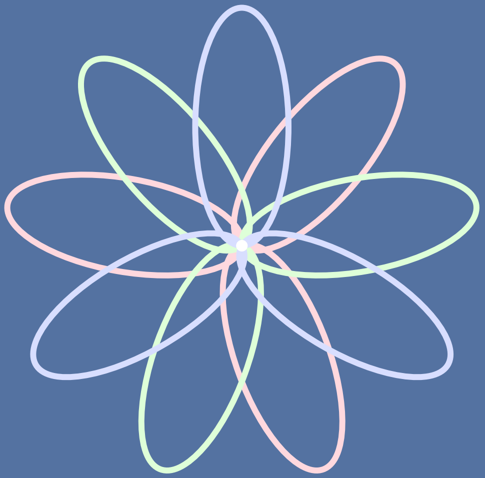
- Let F be the Fermat cubic. Let C be a conic of general form:

$$C : ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

- Select a type-9 point P, write its coordinates as $P = [1 : y_P : z_P]$.
- Put $x = 1$, $y = y + y_P$ and $z = z + z_P$ in F and C.
- *Condition 1*: C has to pass through $(0, 0)$.
- "Remove" monomial z from C.
- *Condition 2*: the coefficient at y in C has to be equal to 0.
- "Remove" monomial yz and z^2 from C.
- *Condition 3*: the coefficient at y^2 in C has to be equal to 0.

Conics and Type-9 Points

- There exists 324 conics with double triple-tangency in these 72 points.
- For each point there are 9 such conics.



Results

For each point of type 9, we considered the pencil of conics passing through it.

- These 9 conics intersect in 30 points (not on Fermat cubic):
  - 3 triple points,
  - 27 double points.
- There are 2016 intersection points overall.
- 1944 points lie on only two conics.
- 72 points lie on nine conics.

The new 72 points lie on $xyz = 0$ (24 points on each line), which is the Hessian of Fermat cubic.

Further results:

- There are 108 conics with double triple-tangency passing through two sextactic points.
- There are 9 such conics passing through each sextactic point.

Further results:

- There are 108 conics with double triple-tangency passing through two sextactic points.
- There are 9 such conics passing through each sextactic point.
- For any line passing through 2 sextactic points S_1 and S_2, its third point of intersection with Fermat cubic is a flex if and only if there exists a double triple-tangency conic passing through S_1 and S_2.

Further results:

- There are 108 conics with double triple-tangency passing through two sextactic points.
- There are 9 such conics passing through each sextactic point.
- For any line passing through 2 sextactic points S_1 and S_2, its third point of intersection with Fermat cubic is a flex if and only if there exists a double triple-tangency conic passing through S_1 and S_2.
- For any line passing through 2 points P_1 and P_2 of type 9, not tangent to Fermat cubic, its third point of intersection with Fermat cubic is a flex if and only if there exists a double triple-tangency conic passing through P_1 and P_2.

Thank You