On closed sets definable in Hensel minimal structures

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Today I will continue my research on Henselian valued fields, and particularly on Hensel minimal structures. The axiomatic theory of those structures was introduced by Cluckers–Halupczok–Rideau (2022). They followed numerous earlier attempts to find suitable approaches in geometry of Henselian valued fields which, likewise o-minimality in real geometry, would realize the postulates of both tame topology and tame model theory. Those attempts have led to various, axiomatically based concepts such as C-minimality, P-minimality, V-minimality, b-minimality, tame structures, and eventually Hensel minimality.

So let K be a 1-h-minimal structure, i.e. a model of a 1-h-minimal theory in an expansion $\mathcal{L}$ of the language of valued fields. The main aim is to establish the following two properties of closed 0-definable subsets A in the affine spaces K^n.

Theorem

Every closed 0-definable subset A of K^n is the zero locus $\mathcal{Z}(g)$ of a continuous 0-definable function g on K^n.

Theorem

For every closed 0-definable subset A of an affine space K^n, there exists a 0-definable retraction $r : X \longrightarrow A$.

The proofs of these theorems make use of a model-theoretic compactness argument and ubiquity of clopen sets in non-Archimedean geometry. Note also that while the former property is a counterpart of the one from o-minimal geometry, the former does not hold in real geometry in general.

As an immediate corollary from Theorem 2, we obtain the following non-Archimedean version of the Tietze–Urysohn extension theorem.

Corollary

Let A be a closed 0-definable subset of an affine space K^n. Then every continuous 0-definable function $f : A \longrightarrow K$ can be extended to a continuous 0-definable function $F : X \longrightarrow K$.

For the proofs of both theorems, we shall proceed by induction with respect to the dimension $k = \dim A$. The case $k = 0$ is straightforward. So assume that the conclusion holds for the subsets of K^n of dimension $< k$ with $1 \leqslant k \leqslant n$.

Lemma

If $A = \bigcup_{i=1}^r A_i$ and the conclusions of the main theorems hold for every subset A_i, then they hold for A.

We shall first consider the case $k < n$, reducing the problem to the sets A of a special form. To this end, we shall apply reparametrized cell decomposition, the above lemma and a model-theoretic compactness argument.

For coordinates $x = (x_1, \ldots, x_n)$ in the affine space K^n, write

$$x = (y, z), \quad y = (x_1, \ldots, x_k), \quad z = (x_{k+1}, \ldots, x_n).$$

Let $\pi : K^n \to K^k$ be the projection onto the first k coordinates. For $y \in K^k$, denote by $A_y \subset K^{n-k}$ the fiber of the set A over the point y.

We can assume that A is the closure of a reparametrized 0-definable cell $C = (C_\xi)_\xi$ of dimension k, with RV-sort parameters ξ and centers c_ξ. Since definable RV-unions of finite sets stay finite, the restriction of π to C has finite fibers of bounded cardinality.

Further, by suitable 0-definable partitioning, we can assume that A is the closure $\overline{E}$ of a 0-definable subset E of dimension k such that all the fibers E_y, $y \in \pi(E)$, have the same cardinality, say s, and the sets $C_j(y) = \{c_{ji}(y),\ i = 1, \ldots, s_j\}$ of j-th coordinates of the fibers E_y have the same cardinality, say s_j, for each $j = k+1, \ldots, n$. Since the fibers E_y are finite, the projection

$$F := \pi(E) \subset K^k$$

is of dimension k. Again this fact and cell decomposition, along with the above Lemma and the induction hypothesis, allow us to come down to the case where F is an open subset of K^k.

Now consider the polynomials

$$P_j(y, Z_j) := \prod_{z \in C_j(y)} (Z_j - z) = \prod_{i=1}^{s_j} (Z_j - c_{ji}(y)), \ \ y \in F, \ j = k+1, \ldots, n,$$

Then

$$P_j(y, Z_j) = Z_j^{s_j} + b_{j,1}(y) Z_j^{s_j - 1} + \ldots + b_{j,s_j}(y), \ \ j = k+1, \ldots, n,$$

where $b_{j,i} : F \to K$, $i = 1, \ldots, s_j$, are 0-definable functions.

We still need the following lemma, which resembles to some extent the primitive element theorem from algebraic geometry.

Lemma

There exist a finite number of linear functions

$$\lambda_l : K^{n-k} \to K, \ \ l = 1, \ldots, p,$$

with integer coefficients such that, for every $y \in F$, λ_l is injective on the product $\prod_{j=k+1}^{n} C_j(y)$ for some $l = 1, \ldots, p$.

Hence we can even assume, after a suitable 0-definable partitioning, that one linear function

$$\lambda : K^{n-k} \to K$$

with integer coefficients is injective on every product

$$\prod_{j=k+1}^{n} C_j(y), \quad y \in F.$$

Consider now the polynomial

$$P(y, Z) := \prod_{z \in E_y} (Z - \lambda(z)) = Z^s + b_1(y) Z^{s-1} + \ldots + b_s(y),$$

where $b_j : F \to K$ are 0-definable functions. Then

$$E = \{x = (y, z) \in F \times K^{n-k} :$$

$$P_{k+1}(x_1, \ldots, x_k, x_{k+1}) = \ldots = P_n(x_1, \ldots, x_k, x_n) =$$

$$P(x_1, \ldots, x_k, \lambda(x_{k+1}, \ldots, x_n)) = 0\}.$$

The sets of all points at which the functions $b_{ji}(y)$ and $b_i(y)$ are not continuous are 0-definable subsets of F of dimension $< k$, and so are the closures of those sets. This along with the induction hypothesis allow us to additionally assume that $b_{ji}(y)$ and $b_i(y)$ are continuous functions on the open subset F. Hence E is a closed subset of $F \times K^{n-k}$, and thus

$$\partial E \subset \partial F \times K^{n-k}.$$

In this manner, we have reduced the proofs of the theorems under study to the case where A is the closure of the set E described above. Moreover, in the proof of the first theorem, we can assume without loss of generality that the set E is bounded.

Sketch of the proof of Theorem 1. Since F is an open subset of K^k, its frontier ∂F is a closed subset of K^k of dimension $< k$. By the induction hypothesis, ∂F is the zero locus of a continuous 0-definable function $f : K^k \to K$.

Observe now that the functions $b_{ji}(y)$ are bounded because so are the sets A and E under study. Therefore the functions

$$f(y) \cdot b_{ji}(y) \text{ and } f(y) \cdot b_i(y)$$

extend by zero through ∂F to continuous functions on $\overline{F}$. And then they extend by zero off F to continuous 0-definable functions on K^k.

We can thus regard the coefficients of the following polynomials (in the indeterminates $Z_{k+1}, \ldots, Z_n$ and Z, respectively):

$$Q_{k+1}(y, Z_j) := f(y) \cdot P_{k+1}(y, Z_{k+1}), \ldots, \; Q_n(y, Z_n) := f(y) \cdot P_n(y, Z_n)$$

and

$$Q(y, Z) := f(y) \cdot P(y, Z),$$

as continuous 0-definable functions on K^k vanishing off the set F.

Put

$$G := \{x \in K^n : \ Q_{k+1}(x_1, \dots, x_k, x_{k+1}) = \dots =$$

$$Q_n(x_1, \dots, x_k, x_n) = Q(x_1, \dots, x_k, \lambda(x_{k+1}, \dots, x_n)) = 0\}.$$

Then

$$G \cap (F \times K^{n-k}) = E \qquad (*)$$

and

$$G \cap ((K^k \setminus F) \times K^{n-k}) = (K^k \setminus F) \times K^{n-k}. \qquad (\dagger)$$

Put

$$\mathcal{E} := \Big\{(b, c, z) \in E \times K^{n-k} : \ b \in F \ \wedge \ \forall\, y \in \partial F \ |z| < |y - b|$$

$$\wedge \ \ \forall\, d \in K^{n-k} \ [((b, d) \in E, \ c \neq d) \Rightarrow |z| < |d - c|] \Big\}$$

and $\widetilde{E} := p(\mathcal{E})$, where

$$p : K^k \times K^{n-k} \times K^{n-k} \ni (y, z, w) \mapsto (y, z + w) \in K^k \times K^{n-k}.$$

Let $\widetilde{A}$ be the closure of $\widetilde{E}$; obviously, $E \subset \widetilde{E}$ and $A \subset \widetilde{A}$.

Remark

Note that the last condition in the definition of the set $\mathcal{E}$ will be used only in the proof of Theorem 2.

It is easy to check that $\widetilde{E} \supset E$ is a clopen subset of $F \times K^{n-k}$, and we have

$$\partial \widetilde{E} = \partial E, \quad \widetilde{A} = \widetilde{E} \cup \partial E$$

and

$$\widetilde{A} \cap ((K^k \setminus F) \times K^{n-k}) = \widetilde{A} \cap (\partial F \times K^{n-k}) = \partial E.$$

Thus eqaulity $\dagger$ yields

$$G \cap \widetilde{A} \cap ((K^k \setminus F) \times K^{n-k}) = \widetilde{A} \cap ((K^k \setminus F) \times K^{n-k}) = \partial E.$$

Further, equality $*$ yields

$$G \cap \widetilde{A} \cap (F \times K^{n-k}) = E \cap \widetilde{A} = E.$$

Combining the above two formulae, we get

$$G \cap \widetilde{A} = E \cup \partial E = A. \qquad (\ddagger)$$

But by the induction hypothesis, ∂E is the zero locus of a continuous 0-definable function $e : K^n \to K$. Therefore the function

$$\widetilde{e}(x) = \begin{cases} 0 & \text{if } \ x \in \widetilde{A}, \\ e(x) & \text{if } \ x \in K^n \setminus \widetilde{A}. \end{cases}$$

is continuous. Obviously, $\widetilde{A}$ is the zero locus of the function $\widetilde{e}$:

$$\widetilde{A} = \{x \in K^n : \ \widetilde{e}(x) = 0\}.$$

Hence and by equality $\ddagger$, we obtain

$$A = \{Q_{k+1}(x_1, \ldots, x_k, x_{k+1}) = \ldots = Q_n(x_1, \ldots, x_k, x_n) =$$

$$Q(x_1, \ldots, x_k, \lambda(x_{k+1}, \ldots, x_n)) = \widetilde{e}(x) = 0\} \subset K^n,$$

which completes the proof.

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