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ON SOME SUBCLASSES OF REGULAR FUNCTIONS WITH FIXED COEFFICIENTS

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In the paper we shall consider a few classes of holomorphic functions with a fixed coefficient or a fixed system of coefficients.

Definition 1. Let C denote the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots \tag{1}$$

which are regular in the disc $E = \{z,\ |z| < 1\}$ and satisfy in E the condition $\operatorname{Re} p(z) > 0$ ([3]).

Accordingly, C_R is a subclass of C of functions of form (1) with real coefficients.

It is well known ([5]) that $q_R \in C_R$ if and only if q_R has the representation

$$q_R(z) = \int_{-\pi}^{\pi} \frac{1 - z^2}{1 - 2z\cos t + z^2} d\mu(t) \tag{2}$$

where $\mu \in M[-\pi, \pi]$, M is the class of functions non-decreasing in the interval $[-\pi, \pi]$, with the normalization $\int_{-\pi}^{\pi} d\mu(t) = 1$.

Definition 2. Let next $P(2\alpha e^{i\Theta}, n, n+k)$, $n, k \in N$, $n \geq 1$, $1 \leq k \leq n$, denote the subclass of C of functions q of the form

$$q(z) = 1 + 2\alpha e^{i\Theta} z^n + p_{n+k} z^{n+k} + \ldots \tag{3}$$

where the coefficient $2\alpha e^{i\Theta}$ at z^n is fixed, $\alpha \in (0, 1)$, $\Theta \in [0, 2\pi)$.

Accordingly, let $P_R(2\alpha, n, n+k)$, denote a subclass of $P(2\alpha e^{i\Theta}, n, n+k)$ of functions q_R of the form

$$q_R(z) = 1 + 2\alpha z^n + 2\alpha_{n+k} z^{n+k} + \dots \tag{4}$$

where all the coefficients are real and the coefficient 2α, $\alpha \in (-1, 1)$ at z^n is fixed.

Note 1. If, in the definition of the class $P(2\alpha e^{i\Theta}, n, n+k)$, the value of Θ is not fixed, the coresponding class will be denoted by $\bar{P}(2\alpha, n, n+k)$ and called a class of functions regular in E with the fixed module 2α of the coefficient at z^n.

Theorem 1 [6]. *A function $q \in P(2\alpha e^{i\Theta}, n, n+k)$ if and only if q has the representation*

$$q(z) = \frac{1 + \alpha e^{i\Theta} z^n + (\alpha e^{-i\Theta} + z^n)\omega(z)}{1 - \alpha e^{i\Theta} z^n + (\alpha e^{-i\Theta} - z^n)\omega(z)}, \qquad z \in E, \tag{5}$$

where the function $\omega(z) = c_k z^k + c_{k+1} z^{k+1} + \ldots \in \Omega(k)$, i.e. ω is regular in E, $|\omega(z)| < 1$ for all $z \in E$, $k \geq 1$ - a positive integer.

Theorem 2. *A function $q_R \in P_R(2\alpha, n, n+k)$ if and only if q_R has the representation*

$$q_R = \frac{1 + \alpha z^n + (\alpha + z^n)\omega_1(z)}{1 - \alpha z^n + (\alpha - z^n)\omega_1(z)} \tag{6}$$

where the function $\omega_1(z) = \alpha_k z^k + \alpha_{k+1} z^{k+1} + \ldots \in \Omega_R(k)$, i.e. ω is regular in E with real coefficients $\alpha_k = \bar{\alpha}_k$ and $|\omega_1(z)| < 1$. In particular, $\omega_1(z)$ may take the form $\omega_1(z) = \frac{1}{2}(\omega(z) + \overline{\omega(\bar{z})})$ where $\omega(z)$ is defined in Theorem 1.

Definition 3. Let next T stand for the class of functions of the form

$$f(z) = z + c_2 z^2 + c_3 z^3 + \ldots, \tag{7}$$

analytic and typically-real in E, i.e. such that $\operatorname{Im} f(z) \cdot \operatorname{Im} z > 0$, $z \in E$, $z \neq \bar{z}$.

Accordingly, we shall denote by $T(c_2)$ a subclass of T of functions (7) where the coefficient $c_2 \in (-2, 2)$ at z^2 is fixed.

As one knows ([7]), $f \in T$ if and only if $C_n = \bar{C}_n$, $n = 2, 3, \ldots$, and $\operatorname{Re}(f(z)\frac{1-z^2}{z}) > 0$ for $z \in E$, which is equivalent to the equality

$$f(z) = \frac{z}{1 - z^2} q_R(z) \tag{8}$$

where $q_R \in C_R$.

Evidently, relation (8) holds only for the functions $f \in T(c_2)$, and $q_R \in P(c_2, 1, 2)$.

Further, of course, formula (6) may be written in the form

$$q(z) = \frac{1 + \alpha z^n + (\alpha + z^n) z^{k-1} \tilde{\omega}_1(z)}{1 - \alpha z^n + (\alpha - z^n) z^{k-1} \tilde{\omega}_1(z)}, \qquad z \in E, \tag{9}$$

where $\tilde{\omega}_1|z| = c_1 z + c_2 z^2 + \ldots \in \Omega_R(1)$.

Between the functions $\tilde{\omega}_1(z) \in \Omega_R(1)$ and the class $P_R(2\alpha, 1, 2)$ there is an evident relationship

$$\omega_1(z) = \frac{q_R(z) - 1}{q_R(z) + 1}. \tag{10}$$

Substituting (10) in (9) we get

$$q_R(z) = \frac{1 + \alpha z^n + (\alpha + z^n) z^{k-1} + [1 + \alpha z^n + (\alpha + z^n) z^{k-1}] q_R(z)}{1 - \alpha z^n - (\alpha - z^n) z^{k-1} + [1 - \alpha z^n + (\alpha - z^n) z^{k-1}] q_R(z)}, \tag{11}$$

$z \in E$.

For the function q_R, formula (11) presents a linear fractional transformation. The determinant Δ of this transformation equals $\Delta = 4(1-\alpha^2) z^{n+k-1} \neq 0$ for $z \in E \backslash \{0\}$.

Now, applying the result of I. Ashnevich and G. Ullina [2], we obtain the following main theorem

Theorem 3. *If $z \in E$ is fixed, then the region of values of the functional $I = q(z)$ in the class $P_R(2\alpha, n, n+k)$ is a convex hull which is bounded by two circles, one of which passes through*

$$q_1 = \frac{1 + \alpha z^n + (\alpha + z^n) z^k}{1 - \alpha z^n + (\alpha - z^n) z^k}, \qquad q_2 = \frac{1 + \alpha z^n - (\alpha + z^n) z^k}{1 - \alpha z^n - (\alpha - z^n) z^k},$$

$$q_3 = \frac{1 + \alpha z^n + (\alpha + z^n) z^{k+1}}{1 - \alpha z^n + (\alpha - z^n) z^{k+1}}$$

and the other passes through the points q_1, q_2 and q_4 where

$$q_4 = \frac{1 + \alpha z^n - (\alpha + z^n) z^{k+1}}{1 - \alpha z^n - (\alpha - z^n) z^{k+1}}.$$

In particular, in the case $n = k = 1$, we obtain the corresponding four points:

$$q_1 = \frac{1 + 2\alpha z + z^2}{1 - z^2}, \qquad q_2 = \frac{1 - z^2}{1 - 2\alpha z + z^2}, \qquad q_3 = \frac{1+z}{1-z} \frac{1 - (1-\alpha) z + z^2}{1 + (1-\alpha) z + z^2},$$

$$q_4 = \frac{1-z}{1+z} \frac{1 + (1+\alpha) z + z^2}{1 - (1+\alpha) z + z^2}.$$

Now, using formula (8), we get

Corollary 1. *The region of values of the functional $I = f(z)$ in the class $T(2\alpha)$ is a convex hull which is bounded by two circles, one of which passes through*

$$f_1 = \frac{z}{1 - 2\alpha z + z^2}, \qquad f_2 = \frac{1 + 2\alpha z + z^2}{(1 - z^2)^2}, \qquad f_3 = \frac{z}{(1 - z^2)^2} \frac{1 - (1-\alpha) z + z^2}{1 + (1-\alpha) z + z^2},$$

and the other - through the points f_1, f_2 and f_4 where

$$f_4 = \frac{z}{(1+z)^2} \frac{1 + (1+\alpha) z + z^2}{1 - (1+\alpha) z + z^2}.$$

Note 2. This result was obtained by means of another method by W.E. Alenitsin [1] and E.G. Golusina [4].

It is interesting that the points f_3 and f_4, obtained by us, differ from the coresponding points in [1] and [4] at the second factor (in [1] and [4] it is equal one). However, it can be shown that the points f_1, f_2, f_3 and $\tilde{f}_3 = \frac{z}{(1-z)^2}$ belong to the same circle.

We consider another particular case: $n = 2$, $k = 2$. Then we have

Corollary 2. *If $z \in E$ is fixed, then the region of values of the functional $I = q(z)$ in the class $P(2\alpha, 2, 4)$ of functions normalized by the expansion $q(z) = 1 + 2\alpha z^2 + 2p_3 z^3 + \dots$, $z \in E$, is a convex hull which is bounded by two circles, one of which passes through*

$$q_1 = \frac{1 + 2\alpha z^2 + z^4}{1 - z^4}, \qquad q_2 = \frac{1 - z^4}{1 - 2\alpha z^2 + z^4}, \qquad q_3 = \frac{1+z}{1-z}\frac{1 + (1+\alpha)z^2 + z^4}{1 + (1-\alpha)z^2 + z^4}$$

and the other - through the points q_1, q_2 and q_4 where

$$q_4 = \frac{1-z}{1+z}\frac{1 + (1+\alpha)z^2 + z^4}{1 + (1-\alpha)z^2 + z^4}.$$

Using formula (8) we get the following

Note 3. One can also obtain the region of values of the functional $I = f(z)$, $0 \neq z \in E$, in a subclass of typically-real functions generated by functions q belonging to the classes of type $P(2\alpha, n, n + k)$.

References

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O pewnej podklasie funkcji regularnych z ustalonymi współczynnikami

Streszczenie. W pracy przedstawiono kilka nowych rezultatów dotyczących własności pewnych klas funkcji holomorficznych w kole $|z| < 1$ z ustalonym współczynnikiem (bądź ustalonym układem współczynników) rozwinię-

cia tych funkcji w szereg potęgowy o środku w punkcie $z = 0$. Zakładamy przy tym, że rozważane funkcje mają wszystkie współczynniki rzeczywiste.

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