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A VERSAL DEFORMATION

Tadeusz Krasiński (Łódź)

1. Introduction

In singularity theory there are many notions of equivalence of isolated singularities. For an isolated singularity f_0 the problem is which singularities (with respect to this equivalence) can be obtained from f_0 by its deformations. This leads to the notion of a versal deformation of f_0 i.e. such one that any other deformation of f_0 can be "induced" from it – details are in Section 2. In the article we prove the existence of a versal deformation for one particular case of equivalence of singularities - biholomorphisms which leave the origin fixed. The reason for this kind of equivalence stems from the fact that we study some numerical invariants of singularities which are computed at the origin. The idea of the proof of existence of a versal deformation is not new but we couldn't find a direct source of such a theorem.

The proof is based on the proof of a similar theorem for another kind of equivalence of singularities, given in Ebeling [E]. The general case of various equivalences can be found in [AGV], [W].

2. Deformations of singularities

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *isolated singularity*. It means f_0 is the germ (or a representant of this germ) of a holomorphic function such that $f_0(0) = 0$, $\nabla f_0(0) := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)(0) = 0$, $\nabla f_0(z) \neq 0$ for all $z = (z_1, \ldots, z_n) \neq 0$ sufficiently close to $0 \in \mathbb{C}^n$.

A *deformation* of f_0 is the germ (or a representant of this germ) of a holomorphic function

$$f = f(z,s) : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$$

such that
1. $f(z,0) = f_0(z)$,
2. $f(0,s) = 0$.

We will treat the deformation f as a holomorphic family (f_s) of isolated singularities because by general theorem (see [GLS], Ch. I, Thm 2.6) f_s has a finite number of critical points in a neighbourhood of 0 for sufficiently small s.

Remark 1. *The above deformation is called by some authors an unfolding.*

Now we define the isomorphism of deformations over the same space of parameters. Two deformations $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ of f_0 are said to be *isomorphic* if there exists a holomorphic map-germ $\psi : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^n, 0)$ such that
1. $\psi(z,0) = z$,
2. $\psi(0,s) = 0$,
3. $g(z,s) = f(\psi(z,s), s)$.

In other words, there exists a holomorphic family of biholomorphisms ψ_s of neighbourhoods of the origin in $\mathbb{C}^n$ which send 0 to 0, $\psi_0 = \mathrm{id}$, and for which $g_s(z) = f_s(\psi_s(z))$.

Let $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ be a deformation of f_0 and $\varphi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ a holomorphic map-germ. The deformation of f_0 *induced from* f *by* φ is defined by

$$g(z,s) := f(z, \varphi(s)).$$

A deformation $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ of f_0 is called *versal* if any deformation of f_0 is equivalent to one induced from f. It means that for any deformation $g : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ of f_0 there exist two holomorphic map-germs

$$\varphi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0),$$
$$\psi : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}^n, 0)$$

such that
1. $\psi(z,0) = z$,
2. $\psi(0,s) = 0$,
3. $g(z,s) = f(\psi(z,s), \varphi(s))$.

Remark 2. *There is also the notion of universal deformation. It is every versal deformation for which the dimension l of the space of parameters is minimal.*

Let $\mathcal{O}^n$ be the ring of germs of holomorphic functions in n-variables and $\mathfrak{m} \subset \mathcal{O}^n$ its unique maximal ideal.

Theorem 1. *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. Let $g_1, \dots, g_k \in \mathcal{O}^n$ be representatives of a basis of the $\mathbb{C}$-vector space $\mathfrak{m}/\mathfrak{m}\cdot(\nabla f_0)$, where (∇f_0)*

denotes the ideal in $\mathcal{O}^n$ *generated by the partial derivatives* $\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}$ *and " $\cdot$ " means the multiplication of ideals. Then* $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ *defined by*

$$f(z,s) := f_0(z) + s_1 g_1(z) + \cdots + s_k g_k(z)$$

is a versal deformation of f_0.

Remark 3. *The* $\mathbb{C}$*-vector space* $\mathfrak{m}/\mathfrak{m}{\cdot}(\nabla f_0)$ *has a finite dimension by the Hilbert Nullstellensatz. Moreover it is easy to prove that*

$$\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}{\cdot}(\nabla f_0) = \dim_{\mathbb{C}} (\mathcal{O}^n/(\nabla f_0)) + n - 1 = \mu(f_0) + n - 1,$$

where $\mu(f_0)$ *is the Milnor number of* f_0.

Remark 4. *In fact, the deformation in Theorem 1 is universal.*

Remark 5. *When we consider the isomorphism of deformations without condition 2. then almost the same reasoning gives that the versal deformation is of the form*

$$f(z,s) := f_0(z) + s_1 g_1(z) + \cdots + s_{\mu(f_0)} g_{\mu(f_0)}(z),$$

where $g_1, \dots, g_{\mu(f_0)} \in \mathcal{O}^n$ *are representatives of a basis of the* $\mathbb{C}$*-vector space* $\mathcal{O}^n/(\nabla f_0)$.

Remark 6. *Now we may explain the reason for which we consider the equivalence of singularities by imposing on biholomorphisms the condition of leaving the origin fixed. In our papers* [Wa], [BKW] *we study the behaviour of the Milnor numbers in holomorphic families (deformations) of singularities. Since in a deformation of an isolated singularity the isolated critical point may split in several ones we have to choose one of them. So it is natural to choose the one at the origin. But it implies an appropriate notion of the equivalence of singularities.*

3. Proof of the theorem

We will prove Theorem 1 in a more general case, not for the deformation given in the theorem but for deformations satisfying a condition, called *infinitesimally versal deformations.* By definition it is a deformation $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ of f_0 such that the germs

$$\left.\frac{\partial f}{\partial s_1}\right|_{s=0}(z), \dots, \left.\frac{\partial f}{\partial s_k}\right|_{s=0}(z)$$

generate (over $\mathbb{C}$) the vector space $\mathfrak{m}/\mathfrak{m}{\cdot}(\nabla f_0)$. Denote these germs by $\dot{f}_{|1}, \dots, \dot{f}_{|k}$, respectively. The deformation f in Theorem 1 is obviously infinitesimally versal because in this case

$$\left.\frac{\partial f}{\partial s_1}\right|_{s=0}(z) = g_1(z), \dots, \left.\frac{\partial f}{\partial s_k}\right|_{s=0}(z) = g_k(z).$$

Theorem 2. *Let* $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be an isolated singularity. Then each infinitesimally versal deformation is versal.*

To prove Theorem 2 we start from the key lemma.

Lemma 1. *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity and $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ an infinitesimally versal deformation of f_0. If $g : (\mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a deformation of f which additionally satisfies the condition*

$$g(0, s, u) \equiv 0 \tag{3.1}$$

then g is a deformation of f_0 which is isomorphic to one induced from f.

Proof. By assumption g is a deformation of f but it is also a deformation of f_0. In fact

$$\begin{aligned} &g(z, 0, 0) \equiv f(z, 0) \equiv f_0(z), \\ &g(0, s, u) \equiv 0 \qquad \text{(by 3.1)}. \end{aligned}$$

We have to show that there exist two holomorphic map-germs

$$\begin{aligned} &\varphi : (\mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}^k, 0), \\ &\psi : (\mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C}, 0) \to (\mathbb{C}^n, 0) \end{aligned}$$

such that:

1. $\psi(z, 0, 0) \equiv z$,
2. $\psi(0, s, u) \equiv 0$,
3. $g(z, s, u) = f(\psi(z, s, u), \varphi(s, u))$.

We have relations between partial derivatives of f and g

$$\begin{aligned} &\left.\frac{\partial g}{\partial z_i}\right|_{\substack{s=0\\u=0}}(z) = \frac{\partial f_0}{\partial z_i}(z), \quad i = 1, \dots, n, \\ &\left.\frac{\partial g}{\partial s_j}\right|_{\substack{s=0\\u=0}}(z) = \left.\frac{\partial f}{\partial s_j}\right|_{s=0}(z) = \dot{f}_{|j}(z), \quad j = 1, \dots, k. \end{aligned}$$

Hence in $\mathcal{O}^{n+k+1}$ we have

$$\begin{aligned} &\frac{\partial g}{\partial z_i}(z, s, u) = \frac{\partial f_0}{\partial z_i}(z) + \widetilde{g}_i(z, s, u), \quad \widetilde{g}_i \in (s, u)\mathcal{O}^{n+k+1}, \\ &\frac{\partial g}{\partial s_j}(z, s, u) = \dot{f}_{|j}(z) + \widetilde{\widetilde{g}}_i(z, s, u), \quad \widetilde{\widetilde{g}}_i \in (s, u)\mathcal{O}^{n+k+1}. \end{aligned}$$

By assumption

$$\mathfrak{m}_n = \mathfrak{m}_n \cdot \left(\frac{\partial f_0}{\partial z_1}, \dots, \frac{\partial f_0}{\partial z_n}\right) + \mathbb{C} \cdot \dot{f}_{|1} + \dots + \mathbb{C} \cdot \dot{f}_{|k}$$

in $\mathcal{O}^n$. Hence arbitrary element $H(z, s, u) \in \mathfrak{m}_{n+k+1}$ can be represented in the form

$$H(z, s, u) = \sum_{i=1}^{n} \alpha_i(z) \frac{\partial g}{\partial z_i}(z, s, u) + \sum_{j=1}^{k} a_j \frac{\partial g}{\partial s_j}(z, s, u) + \widetilde{H}(z, s, u), \tag{3.2}$$

where $\alpha_i \in \mathfrak{m}_n$, $a_j \in \mathbb{C}$, $\widetilde{H} \in (s, u)\mathcal{O}^{n+k+1}$. This implies that the $\mathcal{O}^{n+k+1}$-module

$$M := \mathfrak{m}_{n+k+1}/(\mathfrak{m}_n(\frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_n}))\mathcal{O}^{n+k+1}$$

is finitely generated (by the elements $\frac{\partial g}{\partial s_1}, \ldots, \frac{\partial g}{\partial s_k}, s_1, \ldots, s_k, u$) and we may apply to M the Weierstrass Preparation Theorem for Modules (see e.g. [E], Thm 2.4) to the homomorphism

$$\pi^* : \mathcal{O}^{k+1} \to \mathcal{O}^{n+k+1}$$

induced by the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^k \times \mathbb{C}$. According to this theorem the finitely generated $\mathcal{O}^{n+k+1}$-module M is also finitely generated $\mathcal{O}^{k+1}$-module (via π^*) if and only if $M/\pi^*(\mathfrak{m}_{k+1})M = M/(s,u)M$ is finitely generated over $\mathcal{O}^{k+1}/\mathfrak{m}_{k+1} \cong \mathbb{C}$. By (3.2) the last condition is satisfied because the classes of elements $\frac{\partial g}{\partial s_1}, \ldots, \frac{\partial g}{\partial s_k}$ generate $M/(s,u)M$ over $\mathbb{C}$. Hence M is finitely generated $\mathcal{O}^{k+1}$-module and moreover the classes of $\frac{\partial g}{\partial s_1}, \ldots, \frac{\partial g}{\partial s_k}$ generate M over $\mathcal{O}^{k+1}$. Thus arbitrary element $H(z,s,u) \in \mathfrak{m}_{n+k+1}$ can be represented in the form

$$H(z,s,u) = \sum_{i=1}^{n} \xi_i(z,s,u) \frac{\partial g}{\partial z_i}(z,s,u) + \sum_{j=1}^{k} \eta_j(s,u) \frac{\partial g}{\partial s_j}(z,s,u), \tag{3.3}$$

where $\xi_i \in \mathcal{O}^{n+k+1}$, $\eta_j \in \mathcal{O}^{k+1}$ and $\xi_i(0,s,u) = 0$. We apply this to the element $\frac{\partial g}{\partial u}$. It belongs to $\mathfrak{m}_{n+k+1}$ because by assumption $g(0,0,u) \equiv 0$. So, by (3.3)

$$\frac{\partial g}{\partial u} = \sum_{i=1}^{n} \xi_i(z,s,u) \frac{\partial g}{\partial z_i}(z,s,u) + \sum_{j=1}^{k} \eta_j(s,u) \frac{\partial g}{\partial s_j}(z,s,u),$$

where $\xi_i \in \mathcal{O}^{n+k+1}$, $\eta_j \in \mathcal{O}^{k+1}$ and $\xi_i(0,s,u) = 0$. Taking representatives of the germs ξ_i and η_j we may consider the vector field

$$\Omega(z,s,u) = [\xi_1(z,s,u), \ldots, \xi_n(z,s,u), \eta_1(s,u), \ldots, \eta_k(s,u), -1]$$

in a neighbourhood $\widetilde{U}$ of the origin in $\mathbb{C}^{n+k+1}$. It generates a local one-parameter group of biholomorphisms in $\widetilde{U}$. In particular there exist a neighbourhood $U \subset \widetilde{U}$ of 0 and the unique holomorphic mapping

$$G : K \times U \to \mathbb{C}^{n+k+1}, \qquad K = \{t \in \mathbb{C} : |t| < \varepsilon\} - \text{a small disc}$$

such that:

1. for $t \in K$ the mapping $G_t : U \to G_t(U) \subset \widetilde{U}$ is a biholomorphism of U on the image,
2. $G_0 = \mathrm{id}_U$,
3. for $t, t' \in K$, $a \in U$ such that $t + t' \in K$, $G_{t'}(a) \in U$ we have $G_{t+t'}(a) = G_t(G_{t'}(a))$ (we will not use this fact in the sequel),
4. for $(z_0, s_0, u_0) \in U$ and the curve $X_{(z_0,s_0,u_0)}(t) := G_t(z_0, s_0, u_0)$, $t \in K$, we have

$$X'_{(z_0,s_0,u_0)}(t) = \Omega(X_{(z_0,s_0,u_0)}(t)), \quad t \in K.$$

In other words, the curves $K \ni t \mapsto X_{(z,s,u)}(t) \in \mathbb{C}^{n+k+1}$ are integral curves of the vector field Ω with initial condition $X_{(z,s,u)}(0) = G_0(z,s,u) = (z,s,u)$.

Shrinking U we may assume that $U \subset \{(z,s,u) : |u| < \varepsilon\}$. For fixed $(z_0, s_0, u_0) \in U$ we denote the coordinates of the curve $X_{(z_0,s_0,u_0)}(t)$, $t \in K$, as follows

$$X_{(z_0,s_0,u_0)}(t) =: (z_1(t), \ldots, z_n(t), s_1(t), \ldots, s_k(t), u(t)).$$

By definition of the field Ω we have $u(t) = u_0 - t$ and hence

$$X_{(z_0,s_0,u_0)}(u_0) = (z_1(u_0), \ldots, z_n(u_0), s_1(u_0), \ldots, s_k(u_0), 0).$$

Then for any $(z, s, u) \in U$ we may define two functions ψ and φ by

$$G_u(z, s, u) = X_{(z,s,u)}(u) =: (\psi(z, s, u), \varphi(s, u), 0)$$

in neighbourhoods of $0 \in \mathbb{C}^{n+k+1}$ and $0 \in \mathbb{C}^{k+1}$, respectively. Precisely, ψ and φ are defined by appropriate projections $\psi(z, s, u) = \pi_{1,\ldots,n}(G_u(z, s, u))$ and $\varphi(s, u) = \pi_{n+1,\ldots,n+k}(G_u(z, s, u))$. The function φ does not depend on z because the coordinates $\eta_1(s, u), \ldots, \eta_k(s, u)$ of the vector field X also do not depend on z. Moreover, ψ and φ are holomorphic because $G_u(z, s, u) = G(u, z, s, u)$ is holomorphic. They satisfy the required conditions. In fact, since $G_0(0, 0, 0) = (0, 0, 0)$ then $\psi(0, 0, 0) = 0$ and $\varphi(0, 0) = 0$. Next, $G_0(z, 0, 0) = (z, 0, 0)$. Hence $\psi(z, 0, 0) = z$. Notice that

$$\begin{aligned}\Omega(0, s, u) &= [\xi_1(0, s, u), \ldots, \xi_n(0, s, u), \eta_1(s, u), \ldots, \eta_k(s, u), -1]\\ &= [0, \ldots, 0, \eta_1(s, u), \ldots, \eta_k(s, u), -1].\end{aligned}$$

Hence for integral curves which start at points of the hyperplane $z_1 = 0, \ldots, z_n = 0$ we have

$$X_{(0,s,u)}(t) = (0, \ldots, 0, s_1(t), \ldots, s_k(t), u - t), \quad t \in K,$$

which gives

$$G_u(0, s, u) = X_{(0,s,u)}(u) = (0, \ldots, 0, s_1(u), \ldots, s_k(u), 0).$$

This implies $\psi(0, s, u) = 0$.

At the end let us notice that g is constant on integral curves of Ω. Indeed, for fixed $(z, s, u) \in U$ sufficiently close to 0 we have for arbitrary $t \in K$

$$\begin{aligned}&(g(X_{(z,s,u)}(t)))' =\\ &= \frac{\partial g}{\partial z_1}(X_{(z,s,u)}(t)) \cdot \xi_1(X_{(z,s,u)}(t)) + \cdots + \frac{\partial g}{\partial z_n}(X_{(z,s,u)}(t)) \cdot \xi_n(X_{(z,s,u)}(t))\\ &+ \frac{\partial g}{\partial s_1}(X_{(z,s,u)}(t)) \cdot \eta_1(X_{(z,s,u)}(t)) + \cdots + \frac{\partial g}{\partial s_k}(X_{(z,s,u)}(t) \cdot \eta_k(X_{(z,s,u)}(t))\\ &\qquad - \frac{\partial g}{\partial u}(X_{(z,s,u)}(t)) \equiv 0.\end{aligned}$$

Hence $g(X_{(z,s,u)}(t)) = \text{const}$ for $t \in K$. Then for $(z, s, u) \in U$ sufficiently close to 0 in $\mathbb{C}^{n+k+1}$

$$\begin{aligned}g(z, s, u) &= g(X_{(z,s,u)}(0)) = g(X_{(z,s,u)}(u)) = g(G_u(z, s, u))\\ &= g(\psi(z, s, u), \varphi(s, u), 0) = f(\psi(z, s, u), \varphi(s, u)).\end{aligned}$$

This ends the proof of the lemma. □

Now we may give the proof of Theorem 2.

Proof of Theorem 2. Suppose $f : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ is an infinitesimally versal deformation of f_0 and $g : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ is an arbitrary deformation of f_0. We have to prove that g is isomorphic to a deformation of f_0 induced from f. We consider the following auxiliary deformation h of f_0 defined by

$$h : (\mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0),$$
$$h(z, s, u) := f(z, s) + g(z, u) - f_0(z)$$

and a finite sequence of deformations $h_0, \ldots, h_l$ of f_0 "connecting" f and g, defined by

$$h_i : (\mathbb{C}^n \times \mathbb{C}^k \times \mathbb{C}^i, 0) \to (\mathbb{C}, 0),$$
$$h_i := h|_{\mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^i \times \{0\})},$$
$$h_i(z, s, u_1, \ldots, u_i) := h(z, s, u_1, \ldots, u_i, 0, \ldots, 0).$$

By the definitions of h and h_i we easily see that:

1. $h_0 = f$,
2. $h_l = h$,
3. h_i is an infinitesimally versal deformation of f_0 for each $i = 0, \ldots, l$, because for every i and j we have

$$\left.\frac{\partial h_i}{\partial s_j}\right|_{\substack{s=0\\u=0}}(z) = \left.\frac{\partial f}{\partial s_j}\right|_{s=0}(z).$$

Each pair of deformations (h_i, h_{i+1}) satisfies the assumptions of the lemma. So, by this lemma each h_{i+1} is isomorphic to a deformation induced from h_i. By transitivity we obtain that h_l is isomorphic to a deformation induced from h_0, i.e. h is isomorphic to a deformation induced from f. But g is induced from h by the mapping

$$\varphi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k \times \mathbb{C}^l, 0),$$
$$\varphi(u) := (0, u).$$

Hence again by transitivity we obtain that g is isomorphic to a deformation induced from f. $\square$

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O pewnej wersalnej deformacji

Streszczenie. W artykule dowodzimy istnienia wersalnej deformacji osobliwości izolowanej dla szczególnego rodzaju równoważności osobliwości – biholomorfizmów zachowujących początek układu współrzędnych.

Faculty of Mathematics and Computer Science
University of Łódź
90-238 Łódź, Banacha 22
e-mail: krasinsk@uni.lodz.pl

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