The Łojasiewicz exponent of a non-degenerate surface singularity
Wykładnik Łojasiewicza niezdegenerowanej osobliwości powierzchni

Tadeusz Krasiński
(joint results with Szymon Brzostowski and Grzegorz Oleksik)
University of Łódź

41 Konferencja „Geometria Analityczna i Algebraiczna”
6-10 Stycznia 2020, Łódź

Introduction

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1. $f(z) \in \mathbf{C}\{z_1, \dots, z_n\}$,

2. $\nabla f: (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}^n, 0)$ has an isolated zero at 0.

The Łojasiewicz exponent of a singularity

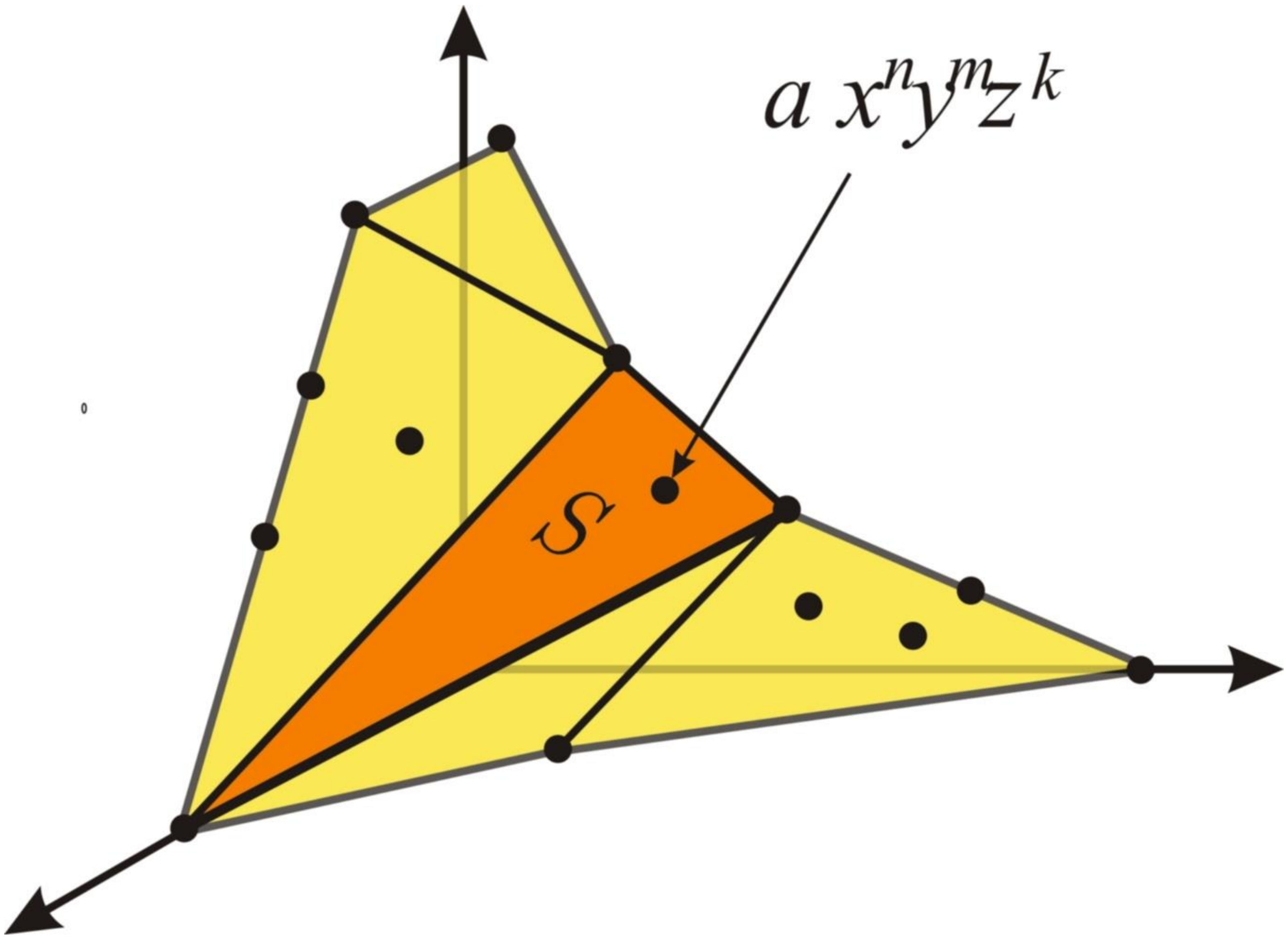
Important invariant of a singularity.

The best exponent $\lambda \in \boldsymbol{R}$ (i.e. infimum) such that the following inequality holds (the Łojasiewicz inequality)

$$\left|\left|\nabla \boldsymbol{f}(\boldsymbol{z})\right|\right| \geq \boldsymbol{c}\ ||\boldsymbol{z}||^{\lambda}$$

in a neighbourhood of the origin in $\boldsymbol{C}^n$.

Newton polyhedron(boundary, diagram) $\Gamma(f)$ of a singularity f



Introduction

Arnold's postulate:

1975-1. Every interesting discrete invariant of a generic singularity with Newton polyhedron is an interesting function of the polyhedron.

The most important example.

The Milnor number of a singularity – the Kushnirenko formula for non-degenerate singularities.

Problem

Give a formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of non-degenerate singularity f in terms of its Newton polyhedron.

The formulas in terms of the Newton diagram

A. **Lenarcik** (1996) A formula for $\mathcal{L}(f)$ in 2 dimensional case (n=2). The singularity depends on two variables $f(x, y)$.

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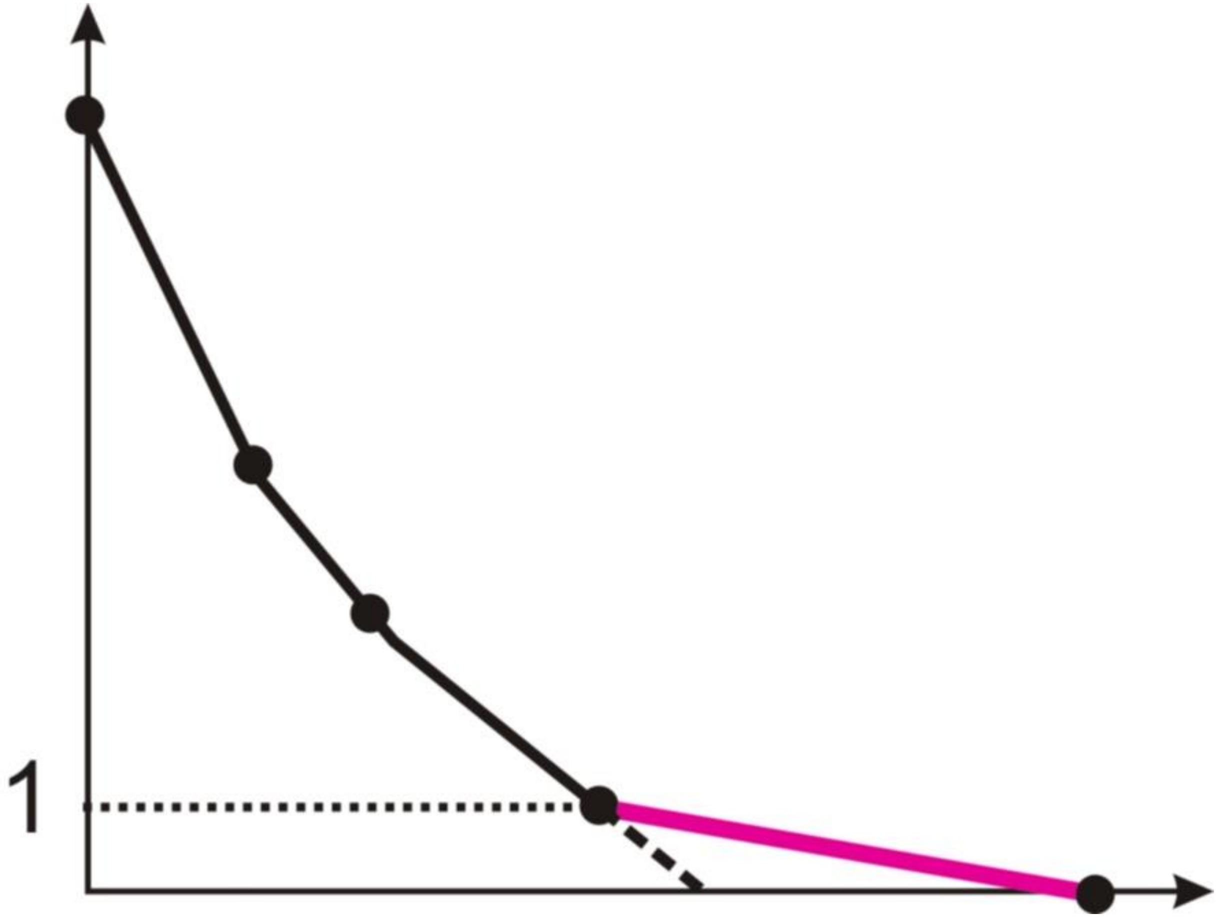
$$\mathcal{L}(f) = \max\left(\alpha(S): S \in \Gamma(f) - E_f\right) - 1$$

E_f - exceptional segments of the Newton diagram $\Gamma(f)$.

The formulas in terms of the Newton diagram

Exceptional segment



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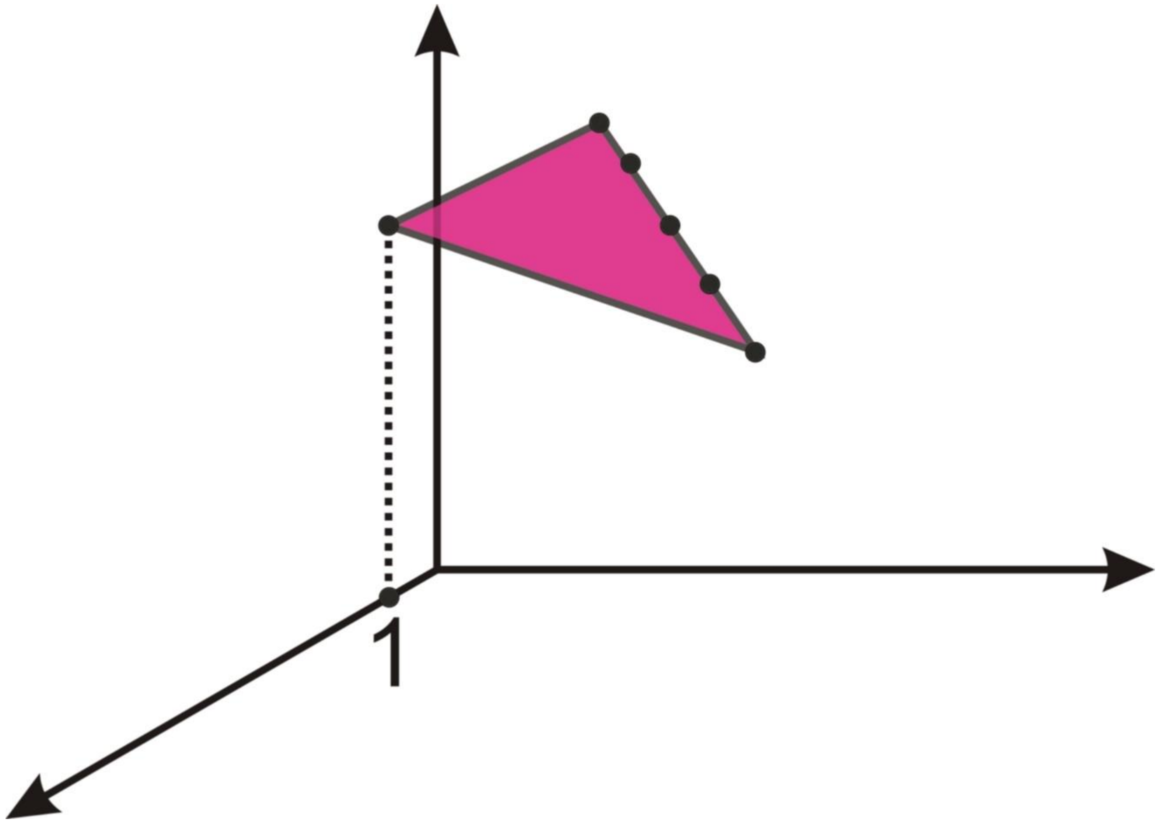
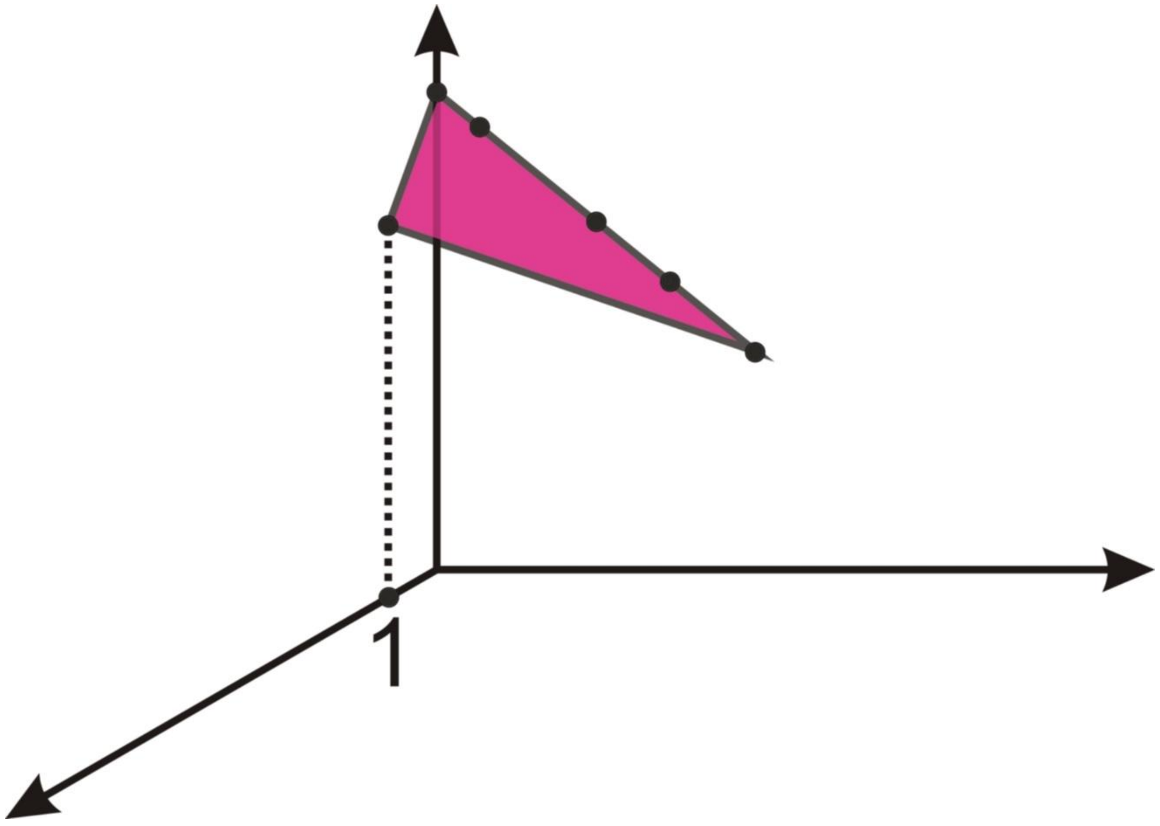
Oleksik definition:
2-dimensionsal face $S \epsilon \Gamma^2(f)$ is said to be exceptional if S has the form

Exceptional faces



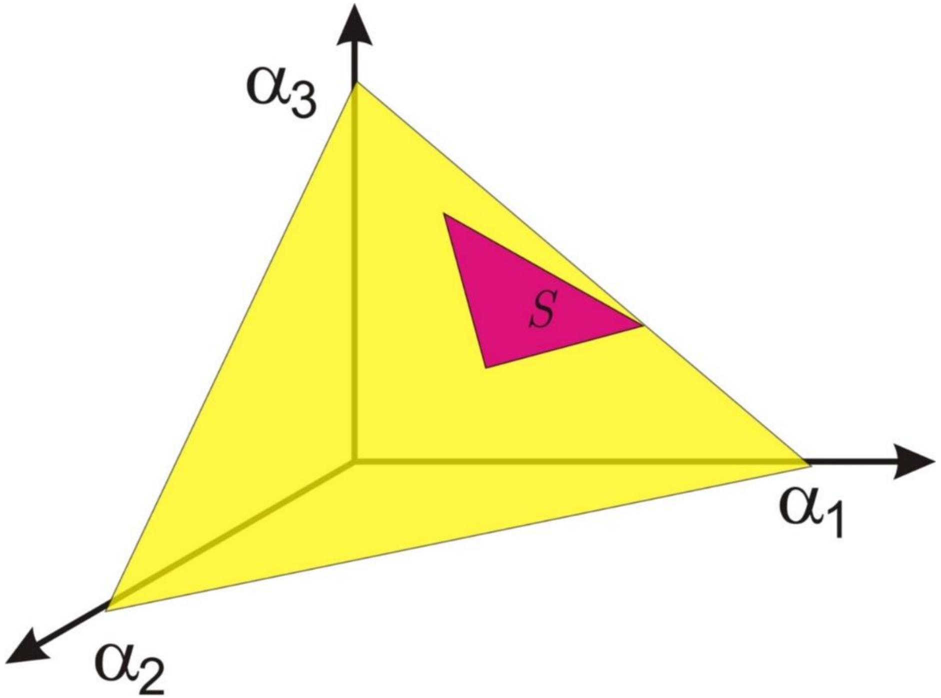


The Oleksik result

Theorem (Oleksik 2010). If $f: (C^3, 0) \to (C, 0)$ is a non-degenerate isolated singularity then

$$\mathcal{L}(f) \leq \max_{S} \left(\alpha(S): S \,\epsilon\, \Gamma^2(f) - E_f\right) - 1.$$

The Oleksik result



$$\alpha(S) = \max(\alpha_1, \alpha_2, \alpha_3)$$

The main result.

Theorem (Brzostowski, Krasiński, Oleksik). If $f\colon (C^3, 0) \to (C, 0)$ is a non-degenerate isolated singularity then

$$\mathcal{L}(f) = \max_{S} \left(\alpha(S) \colon S \in \Gamma(f) - E_f\right) - 1.$$

The main result.

Theorem (Brzostowski, Krasiński, Oleksik). If $f:(C^3,0) \to (C,0)$ is a non-degenerate isolated singularity then

$$\mathcal{L}(f) = \max_{S} \left(\alpha(S): S \in \Gamma(f) - E_f\right) - 1.$$

Notation: $\alpha(f) \coloneqq \max\left(\alpha(S): S \in \Gamma(f) - E_f\right)$

The idea of proof

The inequality

$$\mathcal{L}(f) \leq \alpha(f) - 1$$

- the Oleksik result.

The idea of proof

For inverse inequality

$$\mathcal{L}(f) \geq \alpha(f) - 1$$

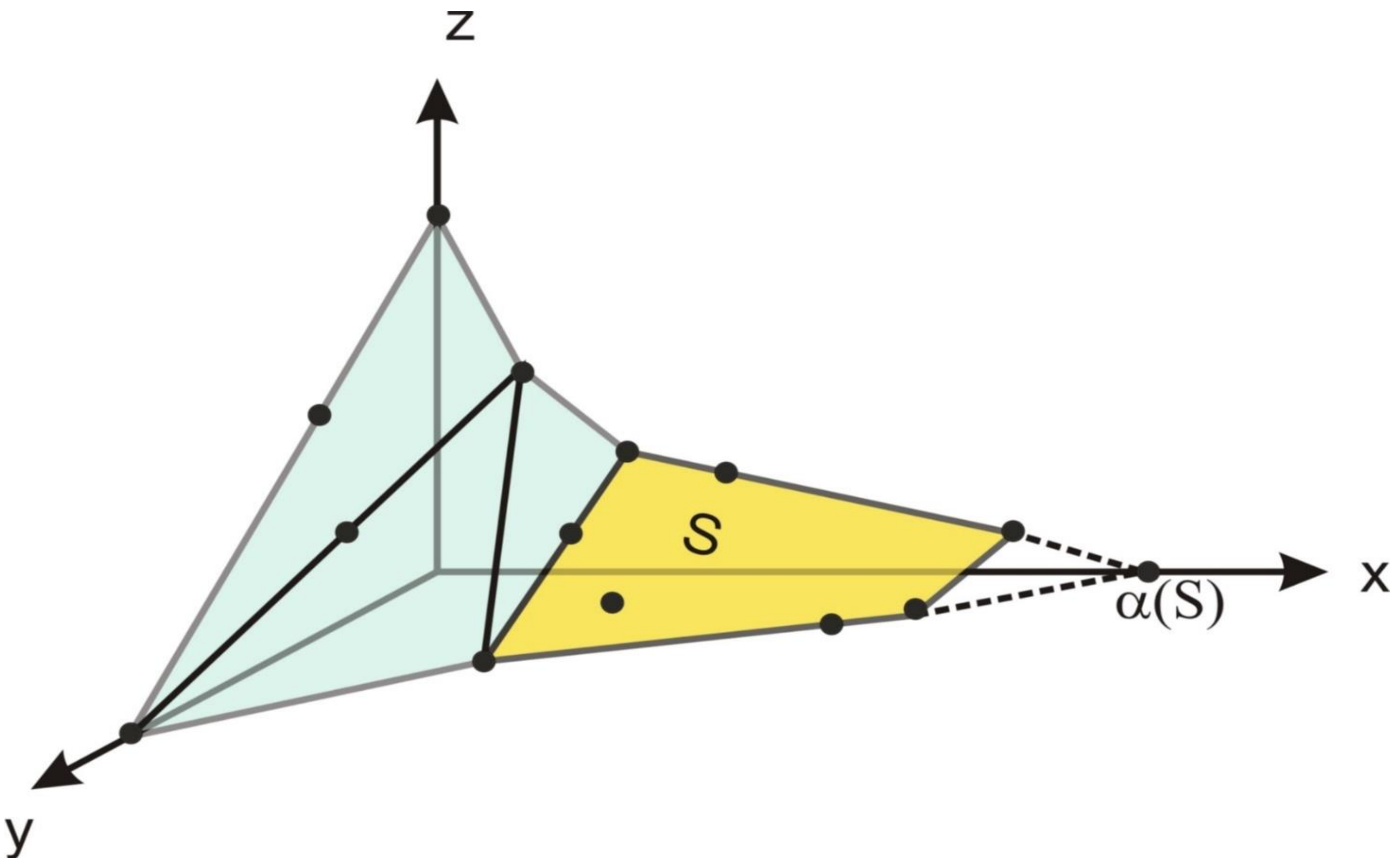
we use known formula for $\mathcal{L}(f)$

$$\mathcal{L}(f) = \max \frac{ord\, \nabla f \circ \varphi}{ord\, \varphi}$$

$\varphi(t) = (x(t), y(t), z(t))$ - a holomorphic curve.

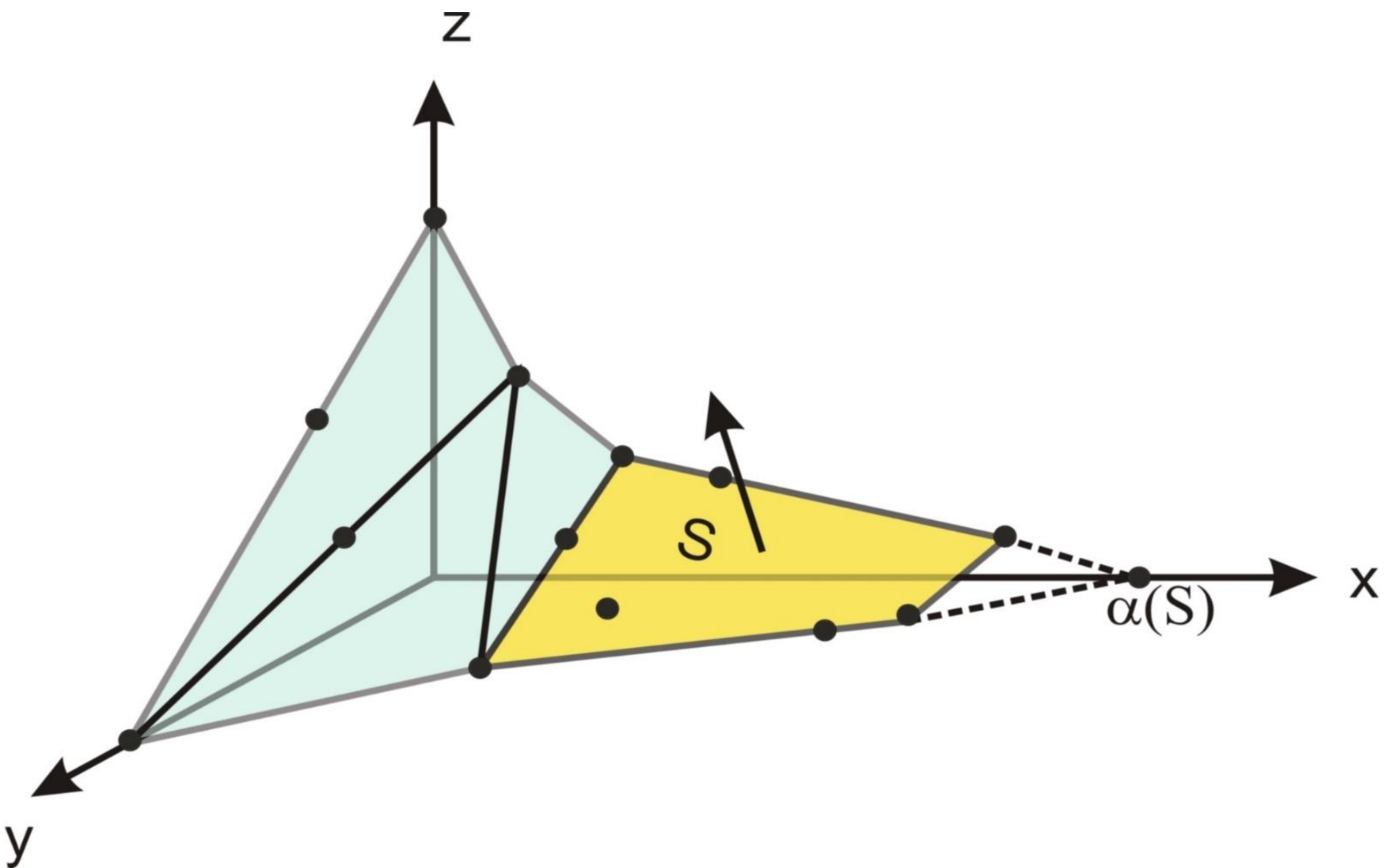
The idea of proof

Let non exceptional face $S \epsilon \Gamma^2(f)$ realize maximum in the definition of $\alpha(f)$. Let this maximum be attained on axis Ox.



The idea of proof

Let $(v, u, w) \in N^3$ be the vector perpendicular to S.



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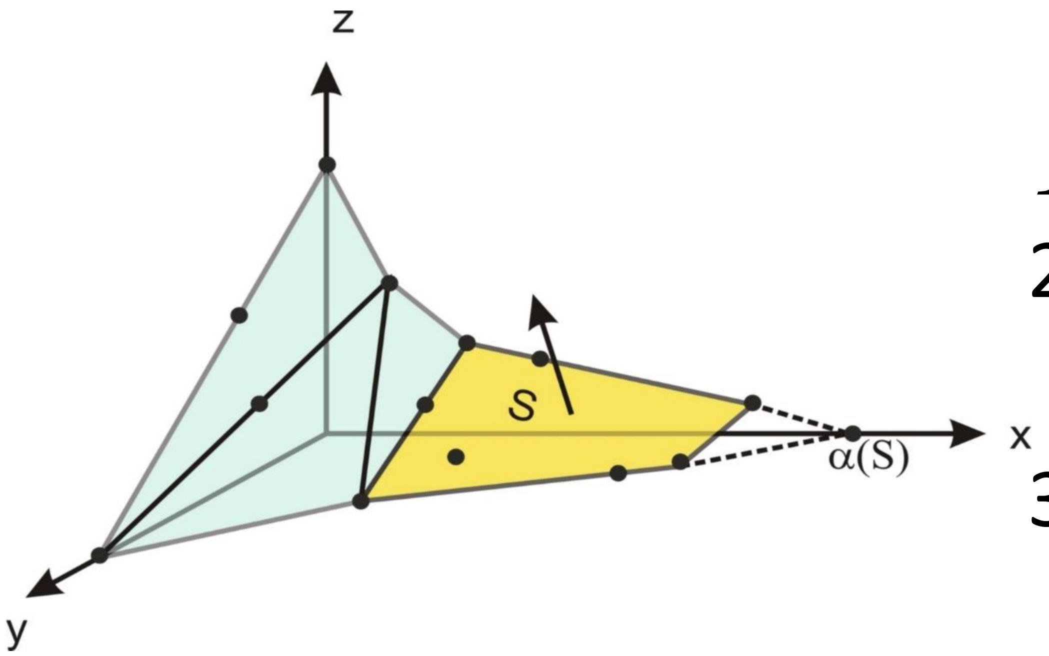

1. $v \leq u, \ v \leq w$,
2. The equation of the plane containing S: $vx + uy + wz = l$,
3. $\alpha(S) = \frac{l}{v}$.

The idea of proof

For monomial curve $\varphi(t) = (at^v, bt^u, ct^w)$ with generic coefficients $a, b, c,$

$$\frac{ord\frac{\partial f}{\partial x} \circ \varphi}{ord\varphi} = \alpha(S) - 1 = \alpha(f) - 1$$

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$$\frac{ord\,\frac{\partial f}{\partial x} \circ \varphi}{ord\varphi} = \alpha(S) - 1 = \alpha(f) - 1$$

Unfortunately

$$\frac{ord\frac{\partial f}{\partial y}\circ\varphi}{ord\varphi} \leq \alpha(f) - 1, \qquad \frac{ord\frac{\partial f}{\partial z}\circ\varphi}{ord\varphi} \leq \alpha(f) - 1$$

The idea of proof

The problem is to do the remaining partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ small enough on the monomial curve φ or on an extension of it $(at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$. The best situation is to find φ such that

$$\frac{\partial f}{\partial y} \circ \varphi \equiv 0, \qquad \frac{\partial f}{\partial z} \circ \varphi \equiv 0$$

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We know

$$\dim V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)=1$$

The idea of proof

We should find parametrization with a given orders of components.

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We apply the classic Maurer theorem (1980) on existence of the parametrization with „a given initial orders of components” of analytic space curves :

The Maurer Theorem

Theorem. If $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ do not generate any element in $C\{x, y, z\}$ with initial $(v, u, w) -$ part being a monomial then there exists a parametrization of a irreducible component of $V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ of the form

$$(at^{kv} + \cdots, bt^{ku} + \cdots, ct^{kw} + \cdots),$$

$$abc \neq 0, k \in \boldsymbol{N}$$

The idea of proof

Condition, in terms of the face S, to fulfill the assumptions of the Maurer Theorem is:

$$\mathrm{mV}(\mathrm{N}(\frac{\partial f_S}{\partial y}(1,y,z)), \mathrm{N}(\frac{\partial f_S}{\partial z}(1,y,z))) > 0$$

Mixed volume of Newton polygons of polynomials $\frac{\partial f_S}{\partial y}(1,y,z)$ and $\frac{\partial f_S}{\partial z}(1,y,z)$ is positive.

The idea of proof

The Newton polygon of a polynomial:

Example. $2 + yz + y^2 + 5y^3z^2 + y^2z^4$

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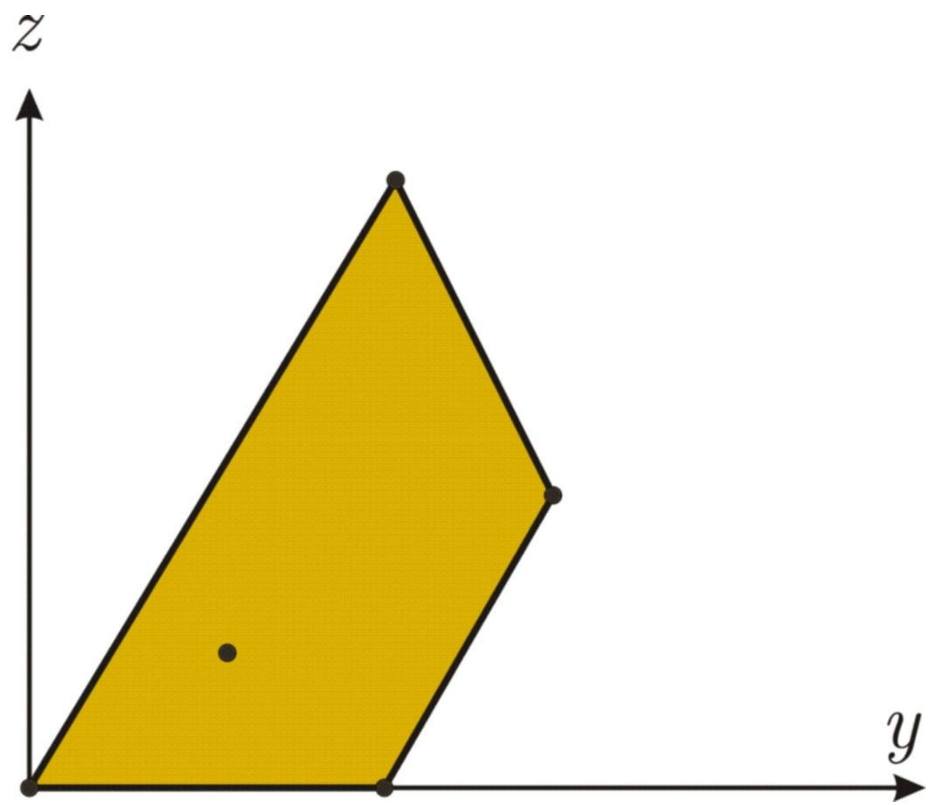

The idea of proof

Mixed volume of Newton polygons of polynomials F, G of two variables

$$mV\big(N(F), N(G)\big) =$$

$$= vol\big(N(F) + N(G)\big) - vol\big(N(F)\big) - vol\big(N(G)\big)$$

The idea of proof

The case

$$\mathrm{mV}(\mathrm{N}(\frac{\partial fs}{\partial y}(1,y,z)),\mathrm{N}(\frac{\partial fs}{\partial z}(1,y,z)))=0$$

holds if and only if polygons $\mathrm{N}(\frac{\partial fs}{\partial y}(1,y,z))$ and $\mathrm{N}(\frac{\partial fs}{\partial z}(1,y,z)$ are parallel segments or one of them is a point.

The idea of proof

The case

$$\mathrm{mV}\left(\mathrm{N}\left(\frac{\partial f_s}{\partial y}(1,y,z)\right), \mathrm{N}\left(\frac{\partial f_s}{\partial z}(1,y,z)\right)\right) = 0$$

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These are particular cases which are consider separately. In each case we find an appropriate parametrization.

The idea of proof

In both cases we use the Brzostowski Theorem

Theorem. In the class of non-degenerate isolated singularities the Łojasiewicz exponent depends only on the Newton diagram.

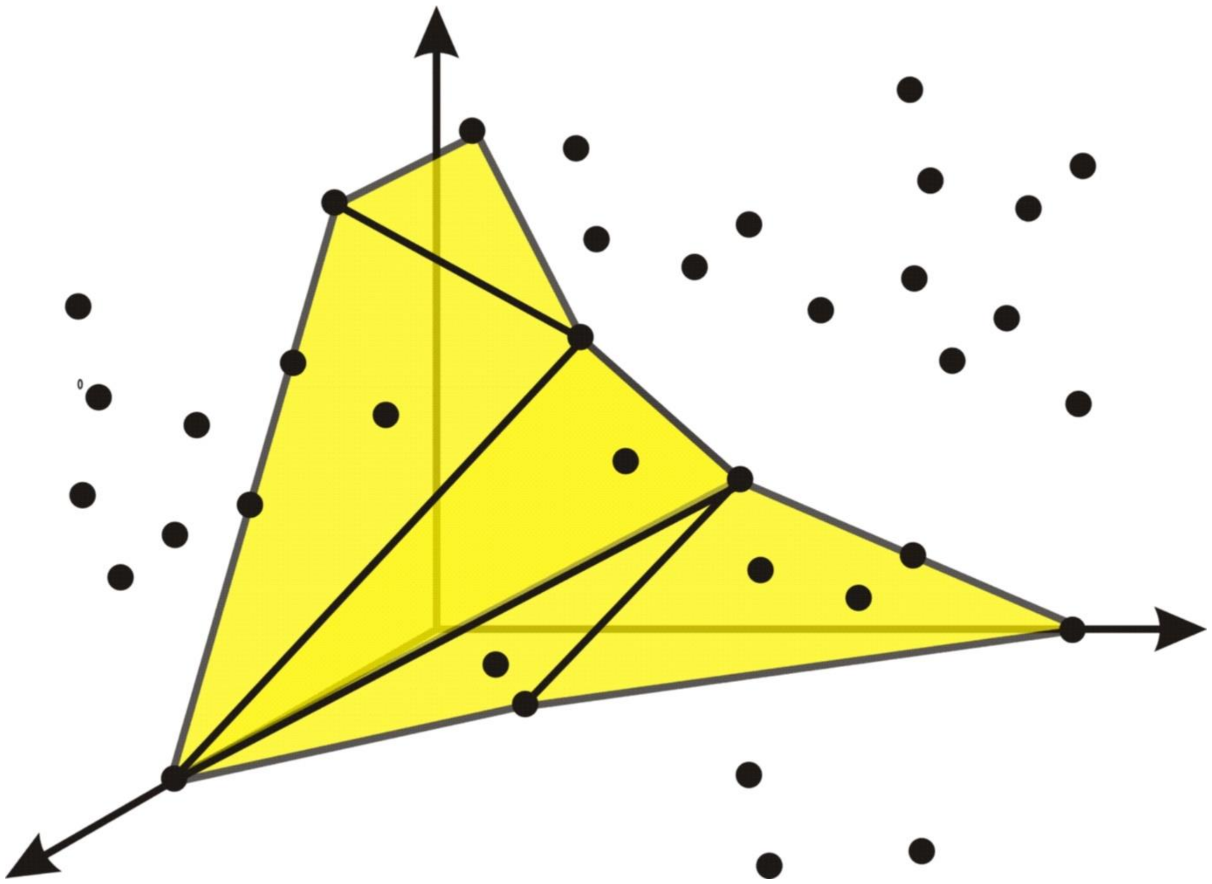
Precisely

If $f, g: (C^n, 0) \to (C, 0)$ are isolated non-degenerate singularities and $\Gamma(f) = \Gamma(g)$ then $\mathcal{L}(f) = \mathcal{L}(g)$.

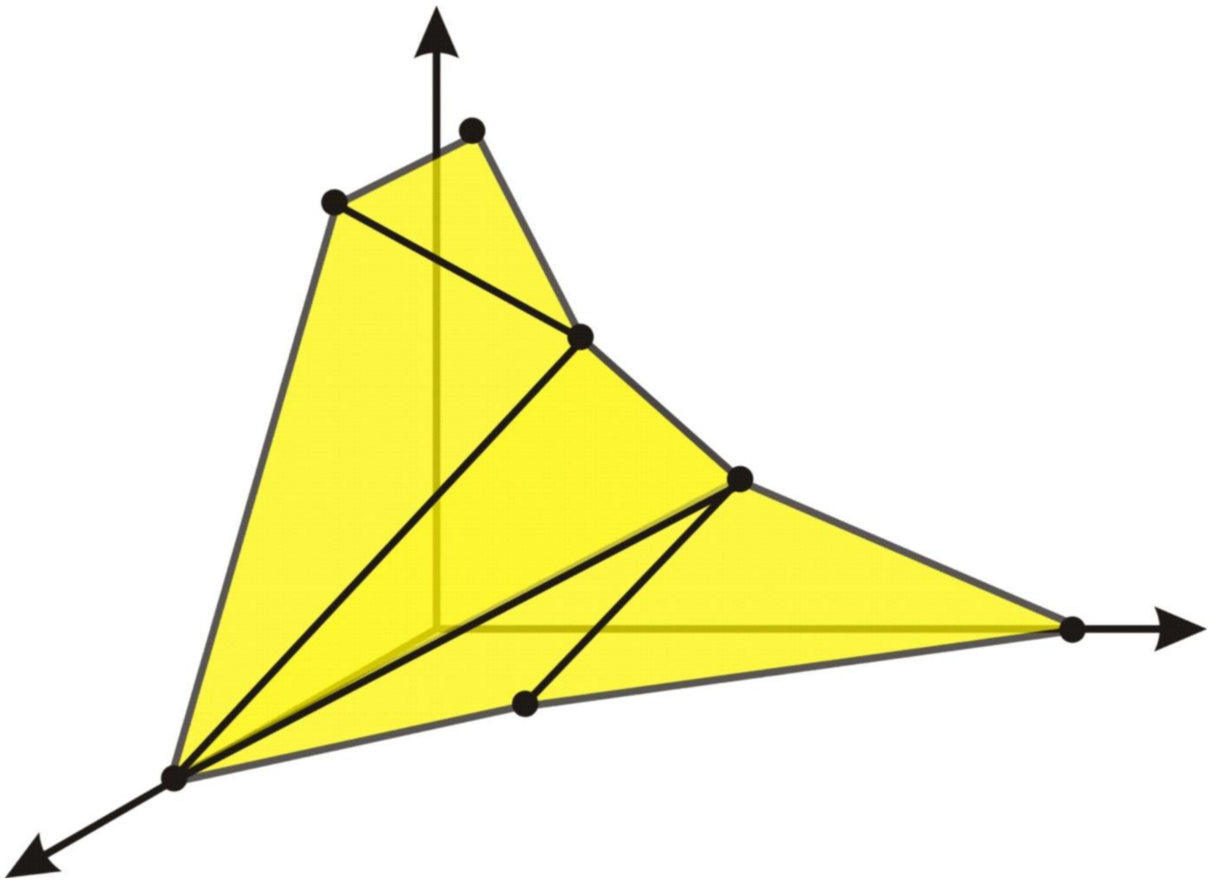
The idea of proof

By this theorem we may replace the initial singularity with another singularity which has the same Newton diagram but with no points above the Newton diagram.

The idea of proof



The idea of proof



Problem

Generalize the result to n-dimensional case.

The end

Thank you.