The Łojasiewicz exponent of the non-degenerate deformations of singularities

Grzegorz Oleksik
(joint results with Szymon Brzostowski and Tadeusz Krasiński)

University of Lodz, Poland

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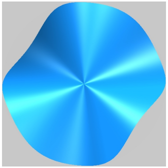

Introduction

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We have two mappings F and G of various domains, classes, fields, etc. such that

$$V(F) \subset V(G)$$

Find (or prove the existence) the best exponent $\alpha \in \mathbb{R}$ such that the following inequality holds (the Łojasiewicz inequality)

$$|F| \geq C|G|^{\alpha}$$

locally or globally.

Introduction

We are interested in the following local, complex variant:

$$F = \operatorname{grad} f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right), \quad G = (z_1, \dots, z_n)$$

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$$|\operatorname{grad} f(z)| \geq C|z|^\alpha$$

Łojasiewicz Exponent

Definition

The Łojasiewicz exponent $\mathcal{L}(f)$ of an isolated singularity

$$f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$$

is the smallest number $\alpha > 0$ such that

$$|\mathrm{grad}\, f(z)| \geq C|z|^\alpha$$

in some neighbourhood of $0 \in \mathbb{C}^n$ and for some $C > 0$.

Lojasiewicz Exponent

Example

- $f(z_1, z_2) := z_1^4 - z_2^3$, $\mathrm{grad}\, f = (4z_1^3, -3z_2^2)$

$$\mathcal{L}(f) = 3,$$

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- $f(z_1, z_2) := z_2^3 + z_2 z_1^3$, $\operatorname{grad} f = (3z_2 z_1^2, 3z_2^2 + z_1^3)$

$$\mathcal{L}(f) = 3\frac{1}{2}.$$

Lojasiewicz Exponent

Theorem (Chang, Lu, Teissier)

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated singularity. Then*

$$Suff(f) = [\mathcal{L}(f)] + 1,$$

$Suff(f)$ *- degree of* C^0*-sufficiency of* f*.*

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The topology of f is determined by its monomials of order at most $Suff(f)$.

It is the smallest integer r such that:
f is topologically equivalent to $f + g$, $\quad \operatorname{ord} g \geq r + 1$.

Łojasiewicz Exponent

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- $|\mathrm{grad}\, f(z)| \geq C|z|^{\mathcal{L}(f)}$ in some neighbourhood of $0 \in \mathbb{C}^n$

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- In which categories is the Łojasiewicz exponent invariant?
- Explain behaviour of the Łojasiewicz exponent in families of singularities.

Lojasiewicz Exponent

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- $n = 2$: exact formula - Chadzyński, García-Barosso,Hà, Krasiński, Kuo-Luo, Lenarcik, Pham, Płoski,

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- Some effective algorithm: Chadzyński and Krasiński ($n = 2$), Płoski, Rodak, Spodzieja ($n \geq 2$).

Lojasiewicz Exponent

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- $n = 2:$ true
- $n = 3:$ true in weighted-homogeneous class
- $n > 2:$ open

Semicontinouity

In general, the Łojasiewicz exponent has no property of semi continuity in families of isolated singularities.

Example

$$f_s = x^2 + sy^2 + y^3,\ s \in \mathbb{C}$$

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$$\mathcal{L}(f_0) = 7 < 9 = \mathcal{L}(f_s)$$

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Theorem

If f_s is $\mu-$constant family of isolated singularities (i.e. the Milnor number is constant in this family) then $\mathcal{L}(f_s)$ is semi continuous from below

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Remark

The Teissier's result was generalized by Płoski (2010) to mappings. (instead of a family of gradient mappings we have a family of mappings with constant multiplicity).

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Remark

For plane curve singularities ($n = 2$) the conjecture is true
Obvious, because $\mu-$ constant family of plane curve singularities is topologically trivial and $\mathcal{L}(f_s)$ is a topological invariant for such singularities.

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$\mathcal{L}(f_s)$ *is constant in* $\mu-$*constant family of* **non-degenerate** isolated **surface** singularities

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Surface singularity:

$$f_s(x,y,z) : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)\,,\; n = 3$$

Family of non degenerate isolated singularities. Each f_s is non-degenerate (in the Kushnirenko sense)

The idea of the proof

The main result follows from 3 other results

Theorem

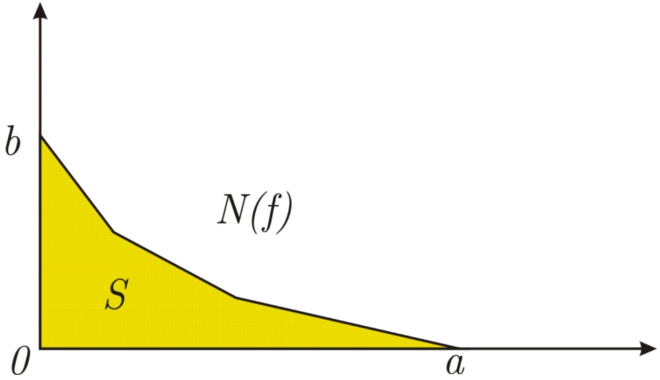
The Kushnirenko result (1976)(n− dimensional). If f is a non-degenerate isolated singularity then

$$\mu(f) = \nu(f),$$

where $\nu(f)$ is the Newton number of f (= effective, discrete invariant which we read off from the Newton polyhedron of f).

From this we get

$$\nu(f_s) = const$$



$$\nu(f) = 2S - a - b + 1$$

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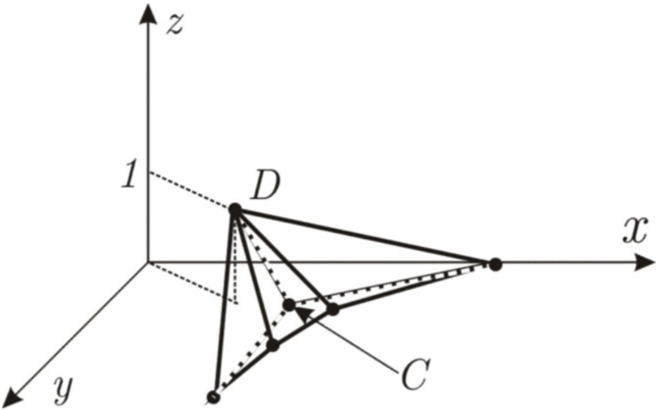
Theorem

Brzostowski, Krasiński, Walewska (2019) (3- dimensional). For two surface singularities f and g if the Newton polyhedrons $N(f)$ and $N(g)$ satisfy $N(f) \subset N(g)$ and $\nu(f) = \nu(g)$ then $N(f)$ and $N(g)$ differ in a very explicit way, they differ on some pyramids with basis in coordinate planes and height one).

From this we get $N(f_s)$ and $N(f_0)$ differ in a very explicit way (because always $N(f_0) \subset N(f_s)$)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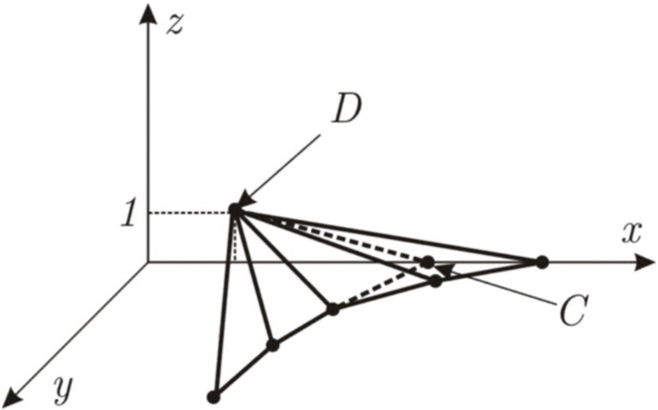
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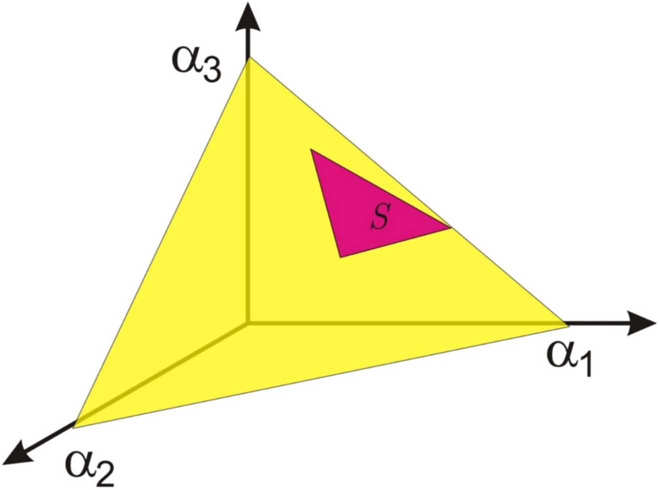
Theorem

Brzostowski, Krasinski, Oleksik (2020 arXiv) (3 dimensional). An effective formula for the Lojasiewicz exponent of a non-degenerate isolated surface singularity f in terms of the Newton polyhedron $N(f)$

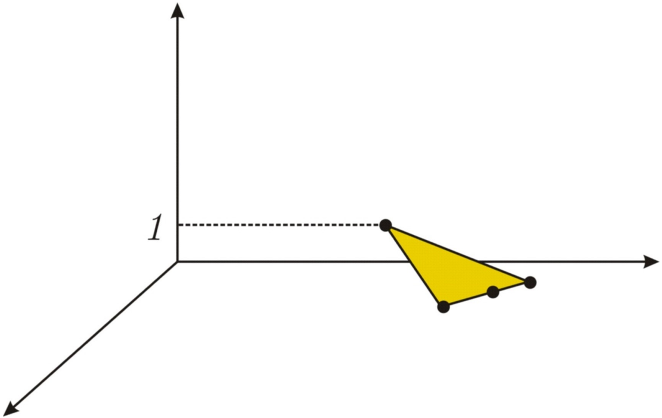
$$\mathcal{L}(f) = \max\{\alpha(S) : S \in \partial N(f) \setminus E(f)\} - 1$$

where $E(f)$ is the set of exceptional faces of $N(f)$.

From this formula follows $\mathcal{L}(f_s) = \mathcal{L}(f_0)$ (because the difference $N(f_s)$ and $N(f_0)$ does not influence on this formula).



$$\alpha(S) = \max(\alpha_1, \alpha_2, \alpha_3)$$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